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**LA THÈSE A ÉTÉ
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**An Analytical Approach to the Least-Square Design of Two-Dimensional
FIR-Digital Filters with Quadrantally Symmetric or
Antisymmetric Frequency Response**

Jie Dong Wang

A Thesis

in

The Department

of

Electrical and Computer Engineering

**Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Engineering at
Concordia University
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ABSTRACT

An Analytical Approach to the Least Square Design of Two-Dimensional FIR Digital Filters with Quadrantally Symmetric or Antisymmetric Frequency Response

Jie Dong Wang

This thesis is concerned with achieving a computationally efficient analytical solution to the least square design problem of two-dimensional real zero-phase FIR digital filters with quadrantally symmetric or antisymmetric frequency response. The problem is investigated by formulating it as a least square approximation resulting in the development of a number of matrices. An in-depth study of these matrices reveals a number of useful properties. These properties are exploited to obtain an optimal analytical solution for the filter coefficients making it unnecessary to use either of the time-consuming methods of optimization, iteration, or matrix inversion.

The process of design solution results in the evolution of a matrix whose elements can be evaluated easily by the knowledge of sampling density of the specified frequency response and the order of the filter to be designed. The coefficients of the designed filter can readily be obtained by carrying out a couple of matrix multiplications involving this matrix and the specified frequency response matrix. It is shown that the computational complexity is greatly reduced because of the reduced order of the matrices involved, their specific characteristics, and the analytical approach. Furthermore, the application of design formulae for the filter design is very simple and efficient.

The design technique is illustrated through examples of quadrantally symmetric and centro-symmetric FIR filters. The results in terms of error in frequency response

compare favourably with those obtained by other techniques. The design time using the proposed technique is significantly smaller than what is required by the l_p -optimization technique, or weighted least square technique employing modified Lawson's algorithm or Harris' ascent algorithm.

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IN MEMORY OF
MY
LOVING MOTHER

TABLE OF CONTENTS

| | |
|---|----|
| LIST OF FIGURES | ix |
| LIST OF TABLES | x |
| LIST OF ABBREVIATIONS AND SYMBOLS | xi |
| CHAPTER I: INTRODUCTION | 1 |
| 1.1 GENERAL | 1 |
| 1.2 MOTIVATION FOR THE STUDY OF FIR FILTERING | 2 |
| 1.3 DESIGN PROBLEM OF 2-D FIR DIGITAL FILTER | 4 |
| 1.4 SOME IMPORTANT TECHNIQUES OF 2-D FIR FILTER DESIGN | 4 |
| 1.5 SCOPE AND ORGANIZATION OF THE THESIS | 6 |
| CHAPTER II: SEPARABLE LEAST SQUARE APPROXIMATION FOR THE DESIGN OF TWO-DIMENSIONAL FIR DIGITAL FILTER | 8 |
| 2.1 INTRODUCTION | 8 |
| 2.2 SYMMETRY AND DECOMPOSITION PROPERTIES IN THE FREQUENCY RESPONSE OF A 2-D REAL ZERO-PHASE FILTER | 8 |
| 2.3 SEPARABILITY AND SYMMETRY IN FILTER DESIGN USING LEAST SQUARE APPROACH | 10 |
| 2.3.1 Quadrantly Symmetric FIR Filters | 10 |
| 2.3.2 Quadrantly Antisymmetric FIR Filters | 14 |
| 2.3.3 Centro-Symmetric FIR Filters | 16 |
| 2.4 CONCLUSION | 16 |
| CHAPTER III: ANALYTICAL SOLUTION FOR THE COEFFICIENT MATRIX AND COMPUTATIONAL CONSIDERATIONS IN ITS EVALUATION | 17 |
| 3.1 INTRODUCTION | 17 |
| 3.2 QUADRANTALLY SYMMETRIC FILTERS | 18 |
| 3.2.1 Properties of the Frequency Response Transform Matrix P | 18 |
| 3.2.2 Properties of Matrix R | 18 |
| 3.2.3 Inversion of Matrix R | 22 |
| 3.2.4 The Coefficient Matrix A | 31 |
| 3.3 QUADRANTALLY ANTISYMMETRIC FILTERS | 39 |
| 3.3.1 Properties of the Frequency Response Transform Matrix Q and Matrix U | 39 |
| 3.3.2 The Coefficient Matrix B | 41 |
| 3.4 CONCLUSION | 42 |

| | |
|--|-----------|
| CHAPTER IV: DESIGN EXAMPLES | 43 |
| 4.1 INTRODUCTION | 43 |
| 4.2 EXAMPLES | 43 |
| 4.3 DISCUSSION AND CONCLUSION | 54 |
| CHAPTER V: CONCLUDING REMARKS | 58 |
| 5.1 CONCLUSIONS | 58 |
| 5.2 SCOPE FOR FUTURE INVESTIGATION | 59 |
| REFERENCES | 60 |
| APPENDIX A | 65 |
| APPENDIX B | 67 |
| APPENDIX C | 68 |

LIST OF FIGURES

| | | |
|----------|---|----|
| Fig. 1.1 | A Linear 2-D FIR Digital-Filter Characterization | 5 |
| Fig. 1.2 | Design Process of a Digital Filter | 5 |
| Fig. 4.1 | Specified Magnitude-Frequency Response of the Fan Filter of Example 4.1 | 44 |
| Fig. 4.2 | Contour Plot of the Magnitude-Frequency Response of the Designed Fan Filter of Example 4.1 | 45 |
| Fig. 4.3 | Perspective Plot of the Magnitude-Frequency Response of the Designed Fan Filter of Example 4.1 | 46 |
| Fig. 4.4 | Magnitude-Frequency Response Specification of the Circularly-Symmetric Lowpass Filter of Example 4.2. | 48 |
| Fig. 4.5 | Contour Plot of the Designed Filter of Example 4.2 with $N = 5$, $M = 32$, $f_c = 0.175$, and $f_s = 0.325$ | 51 |
| Fig. 4.6 | Perspective Plot of the Designed Filter of Example 4.2 with $N = 5$, $M = 32$, $f_c = 0.175$, and $f_s = 0.325$ | 52 |
| Fig. 4.7 | Magnitude-Frequency Response Specification for Example 4.3 | 53 |
| Fig. 4.8 | Contour Plot of the Designed Filter in Example 4.3 | 55 |
| Fig. 4.9 | Perspective Plot of the Designed Filter in Example 4.3 | 56 |

LIST OF TABLE

| | | |
|-----------|--|----|
| Table 3.1 | Elements of the Inverse Frequency Response Transform Matrix S for $N < M$ | 35 |
| Table 3.2 | Expressions for the Functions d_1 , e_1 , f_{e1} , f_{o1} , and f_1 | 36 |
| Table 3.3 | Elements of the Inverse Frequency Response Transform Matrix S for $N = M$ | 37 |
| Table 3.4 | Expressions for the Functions d_2 , e_2 , f_{e2} , f_{o2} , and f_2 | 38 |
| Table 4.1 | A Comparison of Harris' Ascent Algorithm and Modified Lawson's Algorithm with the Proposed Technique in the Design of Two-Dimensional FIR Circularly Symmetric Lowpass Digital Filter | 40 |

LIST OF ABBREVIATIONS AND SYMBOLS

A: Coefficient matrix of quadrantly symmetric FIR filter

B: Coefficient matrix of quadrantly antisymmetric FIR filter

f_c : Passband edge frequency

f_s : Stopband edge frequency

FIR: Finite impulse response

FRTM: Frequency response transform matrix

$h(n_1, n_2)$: Impulse response of a 2-D FIR filter

$H(\omega_1, \omega_2)$: Frequency response of a 2-D FIR filter

$\hat{H}(\omega_1, \omega_2)$: Specified frequency response of a 2-D FIR filter

$H_A(\omega_1, \omega_2)$: Frequency response of an FIR filter possessing quadrantal antisymmetry

$H_S(\omega_1, \omega_2)$: Frequency response of an FIR filter possessing quadrantal symmetry

IFRTM: Inverse frequency response transform matrix

IIR: Infinite impulse response

M: Sampling density of magnitude-frequency response

N: Filter's order

P: Frequency response transform matrix of quadrantly symmetric FIR filter

PB: Passband

Q: Frequency response transform matrix of quadrantly antisymmetric FIR filter

S: Inverse frequency response transform matrix of quadrantly symmetric FIR filter

SB: Stopband

TB: Transition band

V: Inverse frequency response transform matrix of quadrantly antisymmetric FIR filter

WLS: Weighted least square

ω_1 : Horizontal frequency

ω_2 : Vertical frequency

CHAPTER I

INTRODUCTION

1.1 GENERAL

The digital signal processing, a field which has its roots in 17th and 18th century in mathematics and astronomy, has become an important modern tool in a multitude of diverse fields of science and technology [1]-[3]. The classical numerical analysis techniques are used even today in digital signal processing for numerical integration, interpolation, differentiation, and etc. Many of the fundamentals of digital signal processing stem from the classical numerical analysis techniques developed in the 1600s. Important refinements of the techniques in the 1940s and 1950s provide the foundation for digital signal processing. The advent of the digital computer accelerated the development of signal processing technique by making it possible to perform quite rapidly extensive computations on more complex problems. The techniques and applications of the digital signal processing and digital filtering are as old as Newton and Gauss, and as new as digital computers and integrated circuits [1].

Recently, there has been increasing interest in using the two-dimensional (2-D) digital filter for the processing of aerial and satellite photographs, the enhancement of geological data and medical X-rays, and the preprocessing of digital images in pattern-recognition applications [4]. 2-D digital filtering remains a challenging and interesting field of study for several reasons [5]-[8]: many picture processing applications require the use of 2-D signal processing techniques; multidimensional problems generally involve considerably more data than one-dimensional (1-D) ones, as a result, storage requirements and computation time are generally much greater in the multidimensional case; mathematics for handling two-dimensional system is less complete than mathematics for

handling one-dimensional systems and the question of the best approximation becomes very complicated issue unlike the 1-D case where the best approximation is unique; multidimensional systems have many more degrees of freedom, which give a system designer a flexibility not available in the one-dimensional case.

The knowledge of algebra and state-space methods has contributed significantly to the development of new analysis and design techniques in signal processing [9]-[13].

1.2 MOTIVATION FOR THE STUDY OF FIR FILTERING

Two major classes of digital filters are finite impulse response (FIR) and infinite impulse response (IIR) filters. If the impulse response has only a finite number of non-zero samples, the corresponding filter is called an FIR filter. Otherwise, it is known as an IIR filter. We would like to distinguish these two classes of FIR and IIR filters and discuss them as they relate to the filter design problem for certain applications.

FIR filters have the following general properties [7], [14]-[16].

- (i) Linear or real zero-phase designs are easily achieved.
- (ii) Arbitrary frequency response can be readily and closely approximated for sufficiently long impulse responses.
- (iii) The impulse response is always absolutely summable and thus filters are always stable.
- (iv) FIR filters have good quantization properties (i.e., round-off noise can be made small, coefficients can be rounded to reasonable word-lengths for most practical designs, etc.).
- (v) FIR filters can either be implemented directly using a convolution summation or they can be implemented indirectly using discrete transformations.
- (vi) The number of coefficients required for sharp-cutoff filters is generally quite

large. In general, the number of coefficients in an FIR filter can be five to ten times larger than what is in an IIR filter with the same specification

- (vii) Some excellent design techniques of 1-D FIR digital filters are readily extendible to 2-D FIR filters

In contrast, IIR filters have the following properties

- (i) No exact linear phase designs are possible. However, some approximations do exist
- (ii) The stability remains one of the major problems in the design of IIR filters.
- (iii) Most design techniques of 1-D digital filter cannot directly be extended to the 2-D case, because 2-D polynomials usually cannot be factored, whereas 1-D ones can
- (iv) IIR design can be very efficient (small number of poles and zeros), especially for sharp-cutoff frequency response

A major consideration in favor of the use of FIR filter is that in FIR filter the problem of stability does not exist. Furthermore, it can achieve exact linear phase or real zero-phase. This consideration is particularly important in many applications of multidimensional digital signal processing. In image processing, for example, non-zero-phase response tends to destroy lines and edges. A nonlinear phase response tends to disperse those sinusoidal components of a signal that are precisely aligned, such as those that occur at bright spots, lines, and edges [5]. A zero-phase filter provides other rewards as well. Since its frequency response is purely real, the filter design problem is simplified. In addition, the symmetry constraint on the impulse response of the filter can be exploited in the implementation of the filter and also to reduce the number of multiplications needed for its realization [6]. However, for a given filter specification the required order in a 2-D FIR filter can be five to ten times higher than that in a 2-D IIR filter [15]. Thus, in the design of FIR filters, techniques need to be developed which reduce the computa-

tional complexities despite the large order of such filters.

1.3 DESIGN PROBLEM OF 2-D FIR DIGITAL FILTER

In the most general term, a linear 2-D FIR digital filter, as shown in Fig. 1.1, is characterized as a system in which the present output is a linear combination of the present and the past input samples. A frequency-domain design of such a filter is a process in which for a specified frequency response one has to obtain the set of filter coefficients. Ideally, the design process (Fig. 1.2) will generate a filter with a frequency response, say $H(\omega_1, \omega_2)$, that is an exact match to the specified response $\hat{H}(\omega_1, \omega_2)$. However, an exact match is not always possible, so the design technique should generate the best set of coefficients $a(i, j)$'s, giving a response closest to the desired response. Thus, the design becomes a problem of polynomial approximation [16] with the error in approximation and design time being its two important figures of merit.

1.4 SOME IMPORTANT TECHNIQUES OF 2-D FIR FILTER DESIGN

Beginning with the mid seventies, several techniques for the design of 2-D FIR digital filters were advanced [17]-[26]. McClellan [18] and Mersereau *et al* [19] have used transformation on one-dimensional filters in order to design 2-D filters. This technique enjoys a short design time but it is only suboptimal in the Chebyshev sense and cannot be used to approximate closely all magnitude functions [27]. In [20], a 2-D windowing technique has been presented in which a one dimensional window is extended to two-dimensional windows. The technique is simple and has a short design time. However, it does result in the design of a filter which is not optimal in any sense. Most of the design techniques employ linear programming or some iterative procedures, to achieve the design

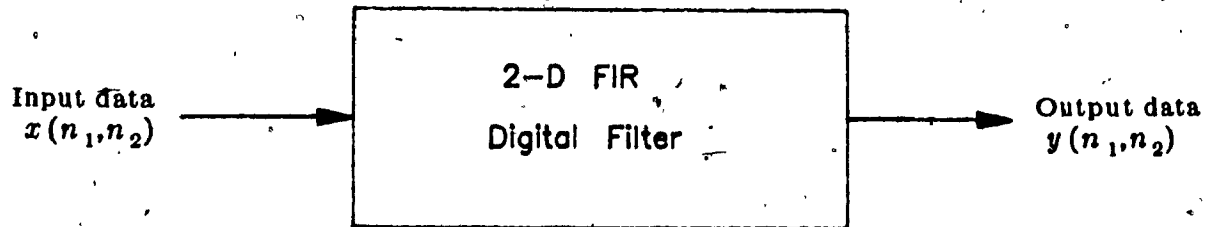


Figure 1.1 A Linear 2-D FIR Digital Filter Characterization.

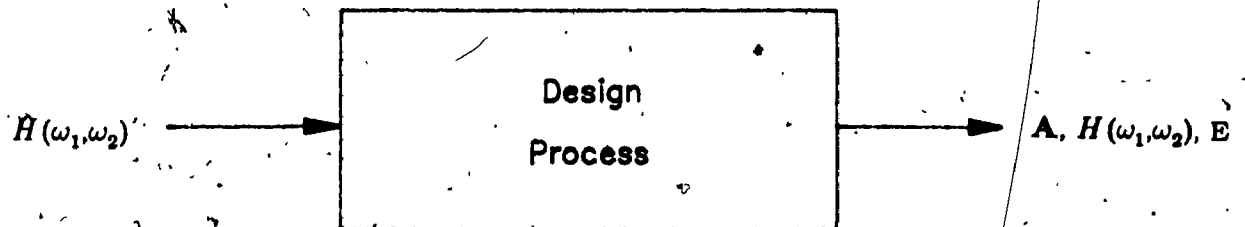


Figure 1.2 Design Process of a Digital Filter.

solution. Linear programming [21] requires a long design time and is restricted for a class of filters only. Iterative techniques [22], although widely used, require long design time as well. Many iterative schemes [28] have been developed to design 2-D FIR filters. More recently, some methods and algorithms [27], [29]-[33], which improved the techniques of FIR filter design, have been advanced. The l_p -optimization technique [27] has a feature that the design time increases only slowly as the number of filter coefficients increases. However, the optimization procedure is rather complex. The design algorithm begins by finding an initial approximation using one of the available windowing techniques. Then, the parameters are optimized using PARTAN [34] requiring the evaluation of gradient vector in each iteration. A weighted least square (WLS) technique presented in [30] has extended the one-dimensional filter design technique by using a formal relation of WLS to Chebyshev approximations and by exploiting this relation to develop an algorithm modifying Lawson's algorithm [35] for the design of almost minimax filters in two dimensions. However, the algorithm lacks a proof of convergence for optimal solution to the minimax design. Windowing and McClellan transformation remain dominant design techniques for FIR filter design [4], [36].

1.5 SCOPE AND ORGANIZATION OF THE THESIS

The objective of this thesis is to achieve an analytical least-square solution to the design problem of real zero-phase 2-D FIR filters with quadrantly symmetric frequency response or quadrantly antisymmetric frequency response. The thesis is mainly concerned with obtaining the filter's coefficients directly from a specified frequency response without a recourse to an optimization technique, iterative procedure or matrix inversion.

In Chapter II, starting with the convolution summation characterization of a 2-D filter with complex exponential excitation, generalized expressions for quadrantly symmetric and antisymmetric frequency response components of a zero-phase FIR filters are

obtained. Based on least-square approximation using linear algebra, an explicit functional relation between the frequency response and the filter's coefficients is derived for each component individually.

In Chapter III, the various matrices introduced in the formulation of the design problem of Chapter II are developed and their properties and characteristics are studied in detail in order to achieve an analytical and computationally efficient technique of obtaining the filter's coefficients. Two important matrices, frequency response transform matrix and inverse frequency response transform matrix, which can be used to obtain designed filter's frequency response and filter's coefficients directly from one another, are introduced.

In Chapter IV, several design examples are considered which illustrate the efficiency of the proposed design technique. The results in terms of error in approximation and design time are compared with other techniques.

Finally, Chapter V concludes the thesis by pointing out the contribution of the proposed investigation and by making suggestions for future work.

CHAPTER II

SEPARABLE LEAST SQUARE APPROXIMATION FOR THE DESIGN OF TWO-DIMENSIONAL FIR DIGITAL FILTERS

2.1 INTRODUCTION

In this chapter two-variable functions that describe the quadrantly symmetric and quadrantly antisymmetric frequency responses of 2-D FIR digital filters are derived. The functions are then used to find a matrix called frequency response transform matrix (FRTM). The FRTM by operating on the coefficient matrix, transforms it into the frequency response matrix of designed filter. It is shown that the least square formulation of the 2-D FIR filter design problem results in the development of a couple of matrices which when operated on the specified frequency response matrix, transforms it into the filter's coefficient matrix.

2.2 SYMMETRY AND DECOMPOSITION PROPERTIES IN THE FREQUENCY RESPONSE OF 2-D REAL ZERO-PHASE FILTER

A 2-D digital filter with an impulse response $h(n_1, n_2)$ and a sinusoidal input

$$x(n_1, n_2) = \exp(j\omega_1 n_1 + j\omega_2 n_2) \quad (2.1)$$

where ω_1 and ω_2 are the horizontal and vertical frequencies, can be characterized by the convolution summation given by

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \exp[j\omega_1(n_1 - k_1) + j\omega_2(n_2 - k_2)] h(k_1, k_2) \quad (2.2)$$

$$= \exp(j\omega_1 n_1 + j\omega_2 n_2) H(\omega_1, \omega_2)$$

where

$$H(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1, n_2) \exp(-j\omega_1 n_1 - j\omega_2 n_2) \quad (2.3)$$

is the filter's frequency response. Since for a real zero-phase filter, $h(n_1, n_2) = h(-n_1, -n_2)$ and for a finite impulse response, $h(n_1, n_2)$ vanishes for $n_1, n_2 > N$, (2.3) can be rewritten as

$$H(\omega_1, \omega_2) = \sum_{n_1=0}^N \sum_{n_2=0}^N 2h(n_1, n_2) \cos(\omega_1 n_1 + \omega_2 n_2) - h(0, 0) \quad (2.4)$$

It is clear from (2.4) that $H(\omega_1, \omega_2)$ possesses centro-symmetry. The right hand side of (2.4) can be expanded giving

$$H(\omega_1, \omega_2) = \sum_{m=0}^N \sum_{n=0}^N a(m, n) (\cos m \omega_1 \cos n \omega_2 - \sin m \omega_1 \sin n \omega_2) \quad (2.5)$$

where

$$a(m, n) = \begin{cases} h(m, n), & \text{for } m=n=0 \\ 2h(m, n), & \text{otherwise} \end{cases} \quad (2.6)$$

It is obvious from (2.5) that $H(\omega_1, \omega_2)$ can be decomposed into two components - one possessing quadrantal symmetry and the other one possessing the corresponding antisymmetry. Thus, the frequency response of a 2-D zero-phase FIR filter possesses centro-symmetry and it can always be decomposed into quadrantly symmetric and antisymmetric components.

If the frequency response of an FIR filter possesses quadrantal symmetry, then it can be characterized by

$$H_S(\omega_1, \omega_2) = \sum_{m=0}^N \sum_{n=0}^N a(m, n) \cos m \omega_1 \cos n \omega_2 \quad (2.7)$$

On the other hand, the frequency response of an FIR filter possessing only quadrantal antisymmetry, is given by

$$\begin{aligned} H_A(\omega_1, \omega_2) &= - \sum_{m=0}^N \sum_{n=0}^N a(m, n) \sin m \omega_1 \sin n \omega_2 \\ &= \sum_{m=0}^N \sum_{n=0}^N b(m, n) \sin m \omega_1 \sin n \omega_2 \end{aligned} \quad (2.8)$$

where

$$b(m, n) = -a(m, n) \quad (2.9)$$

2.3 SEPARABILITY AND SYMMETRY IN FILTER DESIGN USING LEAST SQUARE APPROACH

Designing a 2-D FIR filter is an approximation problem which requires the evaluation of certain independent parameters such that the designed filter's frequency response is optimal in some sense. In this section, we are concerned with the design of FIR filters which possess either quadrantal symmetry or quadrantal antisymmetry in such a way that the least square error between the frequency response of the designed filter and that of the desired filter is minimized.

2.3.1 Quadrantally Symmetric FIR Filters

For a specified quadrantally symmetric frequency response $\hat{H}_S(\omega_1, \omega_2)$, designing an FIR filter consists in finding a total of $(N+1) \times (N+1)$ independent parameters $a(m, n)$ of (2.7) in such a way that the square of error between the designed filter's response, $H_S(\omega_1, \omega_2)$, and the specified response, $\hat{H}_S(\omega_1, \omega_2)$, is minimized. The error square can be written as

$$E = \sum_{k=0}^M \sum_{l=0}^M \left[H_S\left(k\frac{\pi}{M}, l\frac{\pi}{M}\right) - \hat{H}_S\left(k\frac{\pi}{M}, l\frac{\pi}{M}\right) \right]^2 \quad (2.10)$$

where an $(M+1) \times (M+1)$ grid is chosen for the evaluation of error in the first quadrant of $\omega_1 \times \omega_2$ plane. Thus, there are $(M+1) \times (M+1)$ functions of the form $H_S(k\pi/M, l\pi/M)$ and defined by (2.7), to be evaluated. The error defined by (2.10) can be rewritten using matrices as (see Appendix A)

$$\begin{aligned} E &= \text{tr} [(\hat{H}_S - H_S)^T (\hat{H}_S - H_S)] \\ &= \text{tr} [\hat{H}_S^T \hat{H}_S - 2\hat{H}_S^T H_S + H_S^T H_S] \end{aligned} \quad (2.11)$$

where \hat{H}_S and H_S , as given below, are $(M+1) \times (M+1)$ matrices whose elements are the values of frequency response of the desired and designed filters, respectively, at various grid points in $\omega_1 \times \omega_2$ plane.

$$\hat{H}_S = \begin{bmatrix} \hat{H}_S(0,0) & \hat{H}_S(0,\frac{\pi}{M}) & \dots & \hat{H}_S(0,\pi) \\ \hat{H}_S(\frac{\pi}{M},0) & \hat{H}_S(\frac{\pi}{M},\frac{\pi}{M}) & \dots & \hat{H}_S(\frac{\pi}{M},\pi) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{H}_S(\pi,0) & \hat{H}_S(\pi,\frac{\pi}{M}) & \dots & \hat{H}_S(\pi,\pi) \end{bmatrix} \quad (2.12)$$

$$H_S = \begin{bmatrix} H_S(0,0) & H_S(0,\frac{\pi}{M}) & \dots & H_S(0,\pi) \\ H_S(\frac{\pi}{M},0) & H_S(\frac{\pi}{M},\frac{\pi}{M}) & \dots & H_S(\frac{\pi}{M},\pi) \\ \vdots & \vdots & \ddots & \vdots \\ H_S(\pi,0) & H_S(\pi,\frac{\pi}{M}) & \dots & H_S(\pi,\pi) \end{bmatrix} \quad (2.13)$$

Since $H_S(\omega_1, \omega_2)$ given by (2.7) has two separable functions, namely, $\cos \omega_1$ and $\cos \omega_2$, the matrix H_S can be decomposed as

$$H_S = P A T^T \quad (2.14)$$

where A is an $(N+1) \times (N+1)$ coefficient matrix given by

$$\mathbf{A} = \begin{bmatrix} a(0,0) & a(0,1) & \dots & a(0,N) \\ a(1,0) & a(1,1) & \dots & a(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ a(N,0) & a(N,1) & \dots & a(N,N) \end{bmatrix} \quad (2.15)$$

The matrices \mathbf{P} and \mathbf{T} of (2.14) depend, in general, on the two separable functions in ω_1 and ω_2 , respectively. If these two functions are the same, as is the case in our study, and a rectangular sampling in $\omega_1 \times \omega_2$ plane is applied, then $\mathbf{P} = \mathbf{T}$. In that case (2.14) can be rewritten as

$$\mathbf{H}_S = \mathbf{P} \mathbf{A} \mathbf{P}^T \quad (2.16)$$

We will call the $(M+1) \times (N+1)$ matrix \mathbf{P} as the *frequency response transform matrix*, since by the operation given by (2.16), it transforms the coefficient matrix into the filter's frequency response matrix. This matrix, using (2.7), can be expressed as

$$\mathbf{P} = \begin{bmatrix} \cos(0 \times 0 \times \frac{\pi}{M}) & \cos(1 \times 0 \times \frac{\pi}{M}) & \dots & \cos(N \times 0 \times \frac{\pi}{M}) \\ \cos(0 \times 1 \times \frac{\pi}{M}) & \cos(1 \times 1 \times \frac{\pi}{M}) & \dots & \cos(N \times 1 \times \frac{\pi}{M}) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(0 \times M \times \frac{\pi}{M}) & \cos(1 \times M \times \frac{\pi}{M}) & \dots & \cos(N \times M \times \frac{\pi}{M}) \end{bmatrix} \quad (2.17)$$

It is to be noted that the entries in the first row and the first column of \mathbf{P} are all 1's. Substituting (2.16) into (2.11) yields

$$E = \text{tr} [\hat{\mathbf{H}}_S^T \hat{\mathbf{H}}_S - 2\hat{\mathbf{H}}_S^T \mathbf{P} \mathbf{A} \mathbf{P}^T + (\mathbf{P} \mathbf{A} \mathbf{P}^T)^T (\mathbf{P} \mathbf{A} \mathbf{P}^T)] \quad (2.18)$$

Minimization of this quantity is achieved when $\partial E / \partial \mathbf{A} = 0$, which can be used to solve for \mathbf{A} , giving

$$\mathbf{A} = (\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T \hat{\mathbf{H}}_S \mathbf{P} (\mathbf{P}^T \mathbf{P})^{-1} \quad (2.19)$$

For the special case, when $M = N$, \mathbf{P} becomes a square matrix. In this case (2.19) may

be simplified further, giving

$$\mathbf{A} = \mathbf{P}^{-1} \hat{\mathbf{H}}_S (\mathbf{P}^{-1})^T \quad (2.20)$$

In Chapter III, we will show that N cannot be greater than M . Since $\mathbf{P}(\mathbf{P}^T \mathbf{P})^{-1} = ((\mathbf{P}_T \mathbf{P})^{-1} \mathbf{P}^T)^T$, (2.19) can be rewritten as

$$\mathbf{A} = (\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T \hat{\mathbf{H}}_S [(\mathbf{P}^T \mathbf{P})^{-1} \mathbf{P}^T]^T \quad (2.21)$$

Now, define a matrix \mathbf{R} such that

$$\mathbf{R} = \mathbf{P}^T \mathbf{P} \quad (2.22)$$

This is a diagonally symmetric matrix of order $(N+1) \times (N+1)$. Therefore, \mathbf{R}^{-1} is also a diagonally symmetric matrix. Substituting for \mathbf{P} from (2.17) into (2.22), the (n_1, n_2) th element of \mathbf{R} , $R(n_1, n_2)$, is obtained as

$$R(n_1, n_2) = \sum_{k=0}^M \cos(n_1 k \frac{\pi}{M}) \cos(n_2 k \frac{\pi}{M}) \quad (2.23)$$

which is the inner product of the two vectors consisting of n_1 th and n_2 th columns of \mathbf{P} .

Therefore, the complete \mathbf{R} matrix can be expressed as

$$\mathbf{R} = \begin{bmatrix} M+1 & \sum_{k=0}^M \cos(k \frac{\pi}{M}) & \sum_{k=0}^M \cos(2k \frac{\pi}{M}) & \dots & \sum_{k=0}^M \cos(Nk \frac{\pi}{M}) \\ \sum_{k=0}^M \cos(k \frac{\pi}{M}) & \sum_{k=0}^M \cos^2(k \frac{\pi}{M}) & \sum_{k=0}^M \cos(k \frac{\pi}{M}) \cos(2k \frac{\pi}{M}) & \dots & \sum_{k=0}^M \cos(k \frac{\pi}{M}) \cos(Nk \frac{\pi}{M}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^M \cos(Nk \frac{\pi}{M}) & \sum_{k=0}^M \cos(Nk \frac{\pi}{M}) \cos(k \frac{\pi}{M}) & \sum_{k=0}^M \cos(Nk \frac{\pi}{M}) \cos(2k \frac{\pi}{M}) & \dots & \sum_{k=0}^M \cos^2(Nk \frac{\pi}{M}) \end{bmatrix} \quad (2.24)$$

* Note that the elements of \mathbf{R} are denoted as $R(0,0), R(0,1), \dots, R(0,N), \dots, R(N,N)$. Rows and columns are designated as 0th, 1st, 2nd, 3rd, etc. This kind of notation and designation is followed for all matrices throughout the thesis

Substituting (2.22) into (2.21) gives

$$\mathbf{A} = \mathbf{R}^{-1} \mathbf{P}^T \mathbf{H}_s (\mathbf{R}^{-1} \mathbf{P}^T)^T \quad (2.25)$$

Since the separability is applied, the determination of coefficient matrix \mathbf{A} , in general, requires the inversion of only $(N+1) \times (N+1)$ matrix \mathbf{R} and some matrix multiplications as required by (2.22) and (2.25). If separability property of (2.7) is not used for the approximation of the filter, the computations of the coefficients would require the inversion of one symmetric matrix of size $(N+1)(M+1) \times (N+1)(M+1)$ and two matrix-vector multiplications. The matrix-vector multiplication requires an $(N+1)(M+1) \times (N+1)(M+1)$ matrix and an $(M+1)(N+1) \times 1$ vector. Since the symmetry and other properties are applied, the number of operations in the computation of \mathbf{P} , \mathbf{R} , and \mathbf{R}^{-1} is significantly reduced. These points will be elaborated further in the next chapter.

2.3.2 Quadrantly Antisymmetric FIR Filters

Designing a quadrantly antisymmetric FIR Filter consists in finding the independent parameters $b(m, n)$'s of (2.8). However, the coefficients $b(m, n)$'s for $m=0$ or $n=0$, are specified to be zero for the response to be antisymmetric. Thus (2.8) can be rewritten as

$$H_A(\omega_1, \omega_2) = \sum_{m=1}^N \sum_{n=1}^N b(m, n) \sin m \omega_1 \sin n \omega_2 \quad (2.26)$$

As in the case of quadrantly symmetric filters, the frequency response matrix \mathbf{H}_A , using the separability property, can also be decomposed as

$$\mathbf{H}_A = \mathbf{Q} \mathbf{B} \mathbf{Q}^T \quad (2.27)$$

where \mathbf{B} is an $N \times N$ coefficient matrix given by

$$B = \begin{bmatrix} b(1,1) & b(1,2) & \dots & b(1,N) \\ b(2,1) & b(2,2) & \dots & b(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ b(N,1) & b(N,2) & \dots & b(N,N) \end{bmatrix} \quad (2.28)$$

and Q is an $M \times N$ transform matrix of the form

$$Q = \begin{bmatrix} \sin(\frac{\pi}{M}) & \sin(2\frac{\pi}{M}) & \dots & \sin(N\frac{\pi}{M}) \\ \sin(2\frac{\pi}{M}) & \sin(4\frac{\pi}{M}) & \dots & \sin(2N\frac{\pi}{M}) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\pi) & \sin(2\pi) & \dots & \sin(N\pi) \end{bmatrix} \quad (2.29)$$

The coefficient matrix B for the minimum error can be found as

$$B = U^{-1}Q^T \hat{H}_A (U^{-1}Q^T)^T \quad (2.30)$$

where $U = Q^T Q$ is an $N \times N$ diagonally symmetric matrix and the (n_1, n_2) th element of U , $U(n_1, n_2)$, is obtained as

$$U(n_1, n_2) = \sum_{k=1}^M \sin(n_1 k \frac{\pi}{M}) \sin(n_2 k \frac{\pi}{M}) \quad (2.31)$$

Thus, U can be expressed as

$$U = \begin{bmatrix} \sum_{k=1}^M \sin^2(k \frac{\pi}{M}) & \sum_{k=1}^M \sin(k \frac{\pi}{M}) \sin(2k \frac{\pi}{M}) & \dots & \sum_{k=1}^M \sin(k \frac{\pi}{M}) \sin(Nk \frac{\pi}{M}) \\ \sum_{k=1}^M \sin(2k \frac{\pi}{M}) \sin(k \frac{\pi}{M}) & \sum_{k=1}^M \sin^2(2k \frac{\pi}{M}) & \dots & \sum_{k=1}^M \sin(2k \frac{\pi}{M}) \sin(Nk \frac{\pi}{M}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^M \sin(Nk \frac{\pi}{M}) \sin(k \frac{\pi}{M}) & \sum_{k=1}^M \sin(Nk \frac{\pi}{M}) \sin(2k \frac{\pi}{M}) & \dots & \sum_{k=1}^M \sin^2(Nk \frac{\pi}{M}) \end{bmatrix} \quad (2.32)$$

2.3.3 Centro-Symmetric FIR Filters

The design methods for quadrantly symmetric and antisymmetric FIR filters developed in Secs. 2.3.1 and 2.3.2 are not directly applicable to the case of centro-symmetric filters. Therefore, the design of centro-symmetric FIR filters is attempted using symmetrical decomposition. The specified centro-symmetric frequency response is decomposed into quadrantly symmetric and antisymmetric components and two sub-filters are designed independently using the techniques of Secs. 2.3.1 and 2.3.2, respectively. The responses of the two sub-filters are combined in parallel, giving the overall response as

$$H(\omega_1, \omega_2) = \sum_{i=0}^N \sum_{j=0}^N a(i, j) \cos(i\omega_1) \cos(j\omega_2) \pm \sum_{i=1}^{N'} \sum_{j=1}^{N'} b(i, j) \sin(i\omega_1) \sin(j\omega_2) \quad (2.33)$$

where $N \leq M$ and $N' < M$.

It is to be noted that the relationship between the error in the response of the designed centro-symmetric filter and those of the sub-filters is not known and that the use of symmetrical decomposition in the design of such filters may not necessarily guarantee a least-square optimal response. However, it may be worthwhile to check the suitability of the technique through experimental results.

2.4 CONCLUSION

The design problem of zero-phase 2-D FIR filter, with quadrantly symmetric and antisymmetric frequency response has been formulated as a least square approximation problem. An explicit functional relation between the frequency response and the filter coefficients has been obtained by using linear algebra for each of the two types of responses.

CHAPTER III

ANALYTICAL SOLUTION FOR THE COEFFICIENT MATRIX AND COMPUTATIONAL CONSIDERATIONS IN ITS EVALUATION

3.1 INTRODUCTION

This chapter is concerned with obtaining closed-form analytical expressions for the elements of a matrix called the inverse frequency response transform matrix. This matrix is used with the specified frequency response to obtain the filter coefficients directly without a recourse to an optimization technique, iterative procedure or matrix inversion.

The success of the filter design method, introduced in the previous chapter depends on the efficiency with which the coefficient matrices A and B can be evaluated. As seen from (2.25) and (2.30), evaluation of A and B involves a number of matrices. It is noted that the size of the transform matrices P and Q increases linearly as N and M increase and the size of R or U is proportional to N^2 . Computations involved in the coefficient matrix evaluation will, in general, increase considerably as the size of these matrices increases. It is also essential that the matrices R and U be nonsingular and well-conditioned. This chapter, therefore, will first study the properties of matrices P and Q , and R and U , then demonstrate how they can be used to simplify and reduce the computations significantly. A heavy emphasis is placed on the analytical determination of the inverse of matrix R . Using R^{-1} a set of formulae are derived from which, for the given values of M and N , and the specified quadrantly symmetric frequency response, the filter's coefficients can be evaluated. The expressions for the coefficients of quadrantly antisymmetric filters are also derived.

3.2 QUADRANTALLY SYMMETRIC FILTERS

In this section, the design of FIR filters which possess quadrantal symmetry is considered. The properties of matrices P and R are investigated in detail. Finally, a set of very simple formulae for the filter design are derived.

3.2.1 Properties of the Frequency Response Transform Matrix P

It can be easily seen from (2.17) that the elements of the P matrix are ordered structure. The elements of its rows are all shifted copies of the same periodic sequence. Although the order of the matrix P is $(M+1) \times (N+1)$, there are only $M+1$ different elements in the entire matrix and a different subset of these elements appears in each column. The value of an element of P is given by $\cos(i \frac{\pi}{M})$, where depending on the position of the element, its value can be obtained by choosing the value of i in $\{0, 1, 2, \dots, M-1\}$. Specifically, the value of the element in the n th row and k th column, is given by $\cos(n \times k \times \frac{\pi}{M})$. With this kind of matrix, computations increase slowly as N increases, thus making the design of higher-order 2-D FIR filters relatively simpler.

3.2.2 Properties of Matrix R

The most critical point in the development of the proposed design is the inversion of the matrix R as required by (2.25). Generally speaking, matrix inversion is one of the most time-consuming operations, particularly if the order of the matrix is large. We will study the nature of the element $R(n_1, n_2)$ of R for the following four cases.

Case (i): $n_1 = n_2 = 0$ or $n_1 = n_2 = N = M$

For $n_1 = n_2 = 0$, (2.23) can be rewritten as

$$R(0,0) = \sum_{k=0}^M \cos^2(0 \times k \frac{\pi}{M}) = \sum_{k=0}^M 1 = M+1$$

and for $n_1 = n_2 = N = M$,

$$R(N,N) = \sum_{k=0}^M \cos^2(M \times k \frac{\pi}{M}) = \sum_{k=0}^M \cos^2(k\pi) = \sum_{k=0}^M 1 = M+1$$

Case (ii): $n_1 = n_2$ ($n_1 = n_2 \neq 0$ or $n_1 = n_2 = N \neq M$)

In this case, (2.23) can be written as

$$\begin{aligned} R(n_1, n_2) &= \sum_{k=0}^M \cos^2(n_1 k \frac{\pi}{M}) \\ &= \frac{M+1}{2} + \frac{1}{2} \sum_{k=0}^M \cos(2n_1 k \frac{\pi}{M}) \end{aligned} \quad (3.1)$$

Using (B.1), the second term on the right side of (3.1) can be rewritten as

$$\begin{aligned} \sum_{k=0}^M \cos(2n_1 k \frac{\pi}{M}) &= \frac{\cos(2n_1 \frac{\pi}{2}) \sin(\frac{M+1}{2M} 2n_1 \pi)}{\sin(2n_1 \frac{\pi}{2M})} \\ &= \frac{\cos(n_1 \pi) \sin(n_1 \pi + \frac{n_1 \pi}{M})}{\sin(n_1 \frac{\pi}{M})} \\ &= \frac{(-1)^{n_1} [\sin(n_1 \pi) \cos(n_1 \frac{\pi}{M}) + \cos(n_1 \pi) \sin(n_1 \frac{\pi}{M})]}{\sin(n_1 \frac{\pi}{M})} \\ &= (-1)^{2n_1} = 1 \end{aligned}$$

Thus,

$$R(n_1, n_2) = \frac{M}{2} + 1$$

Case(iii): $n_1 \neq n_2$ and $(n_1 + n_2)$ is even

This case implies that n_1 and n_2 are both either even or odd. Thus $(n_1 - n_2)$ will be even. From (2.23),

$$R(n_1, n_2) = \frac{1}{2} \sum_{k=0}^M \cos(ak \frac{\pi}{M}) + \frac{1}{2} \sum_{k=0}^M \cos(bk \frac{\pi}{M}) \quad (3.2)$$

where $a = (n_1 + n_2)$ and $b = (n_1 - n_2)$ are even. Using (B.1), the first summation of the right side of (3.2) can be expressed as

$$\begin{aligned} \sum_{k=0}^M \cos(ak \frac{\pi}{M}) &= \frac{\cos(\frac{a\pi}{2}) \sin(\frac{M+1}{2M} a \pi)}{\sin(\frac{a\pi}{2M})} \\ &= \frac{(-1)^{a/2} \sin(\frac{a\pi}{2} + \frac{a\pi}{2M})}{\sin(\frac{a\pi}{2M})} \\ &= \frac{(-1)^{a/2} [\sin(\frac{a\pi}{2}) \cos(\frac{a\pi}{2M}) + \cos(\frac{a\pi}{2}) \sin(\frac{a\pi}{2M})]}{\sin(\frac{a\pi}{2M})} \end{aligned} \quad (3.3)$$

since a is even, $\sin(\frac{a\pi}{2}) = 0$. Thus,

$$\begin{aligned} \sum_{k=0}^M \cos(ak \frac{\pi}{M}) &= (-1)^{a/2} \cos(\frac{a\pi}{2M}) = (-1)^{a/2} (-1)^{a/2} \\ &= (-1)^a = 1 \end{aligned} \quad (3.4)$$

Similarly, it can be shown that

$$\sum_{k=0}^M \cos(bk \frac{\pi}{M}) = 1 \quad (3.5)$$

From (3.2), (3.4), and (3.5),

$$R(n_1, n_2) = 1$$

Case (iv): (n_1+n_2) is odd

This case implies that both n_1 and n_2 cannot be even or odd simultaneously. Thus (n_1-n_2) is also odd. Again, $R(n_1, n_2)$ can be expressed as in (3.2) where $a = (n_1+n_2)$ and $b = (n_1-n_2)$, in this case, are odd. Thus, $\cos(a\frac{\pi}{2})=0$ and from (3.3),

$$\sum_{k=0}^M \cos(ak\frac{\pi}{M}) = 0 \quad (3.6)$$

Following a similar reasoning,

$$\sum_{k=0}^M \cos(bk\frac{\pi}{M}) = 0 \quad (3.7)$$

Therefore, from (3.2), (3.6), and (3.7),

$$R(n_1, n_2) = 0$$

The above analysis greatly simplifies the matrix R which now takes the following form

$$R = \begin{bmatrix} M+1 & 0 & 1 & 0 & \dots & 0 \\ 0 & \frac{M}{2}+1 & 0 & 1 & \dots & 0 \\ 1 & 0 & \frac{M}{2}+1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \frac{M}{2}+1 \end{bmatrix} \quad (3.8)$$

It should be mentioned that if $N=M$, $R(N, N)=M+1$. From (3.8) it can easily be established that for $N \leq M$

$$|R(n_1, n_1)| > \sum_{\substack{n_2=0 \\ n_2 \neq n_1}}^M |R(n_1, n_2)| \quad (3.9)$$

Thus, R is a strictly diagonally-dominant matrix [37]. As a result R is also nonsingular.

Further, the nondiagonal elements are either zero or one and they interlace one another along each row and column of the matrix.

Before proceeding further, it should be illustrated that N cannot be chosen to be greater than M . Let $N=2M$ and $n_1=0, n_2=N$. Then from (2.23),

$$R(n_1, n_2) = \sum_{k=0}^M \cos 0 \cos(2k\pi) = M+1$$

Obviously, in this case, the condition given by (3.9) is not satisfied. Therefore, matrix R will ill-behave and the method of design will not work. In actual practice, however, N is chosen to be smaller than or equal M and ill condition does not occur.

3.2.3 Inversion of Matrix R

At this stage an expression for the coefficient matrix A , as given by (2.25), has been obtained in terms of R , P and H_S . Even though the elements of matrix R are very much simplified, the expression for A still involves the inversion of R . We will now focus our attention to this inversion problem. Let R be written as

$$R = C+D \tag{3.10}$$

where C is an $(N+1) \times (N+1)$ diagonal matrix expressed as

$$C = \begin{bmatrix} M & & \\ & \frac{M}{2} & \bigcirc \\ & \bigcirc & \frac{M}{2} \end{bmatrix} \tag{3.11}$$

and D as expressed by

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & \dots \\ 0 & 1 & 0 & 1 & \dots & \dots \\ 1 & 0 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3.12)$$

is an $(N+1) \times (N+1)$ singular matrix with a rank of 2. Note that $C(N, N) = M$, when $N = M$. The matrix D can further be decomposed as EE^T , allowing (3.10) to be rewritten as

$$R = C + EE^T \quad (3.13)$$

where E is an $(N+1) \times 2$ matrix, given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \quad (3.14)$$

Using (3.13) and (C.1), the inverse of matrix R can be expanded as

$$R^{-1} = C^{-1} - C^{-1}E(I + E^T C^{-1}E)^{-1}E^T C^{-1} \quad (3.15)$$

in which I is a 2×2 identity matrix. Since C is a diagonal matrix, its inverse can readily be written as

$$C^{-1} = \begin{bmatrix} \frac{1}{M} & & & \\ & \frac{2}{M} & & \\ & & \bigcirc & \\ & & & \frac{2}{M} \end{bmatrix} \quad (3.16)$$

Again, if $N = M$, $C^{-1}(N, N) = \frac{1}{M}$. In order to find an analytical form for the inverse of

R , the following four cases will be studied.

Case (i): $N < M$ and N is odd

When N is odd, from (3.14) and (3.16),

$$I + E^T C^{-1} E = \begin{bmatrix} \frac{M+N}{M} & 0 \\ 0 & \frac{M+N+1}{M} \end{bmatrix} \quad (3.17)$$

Inverse of the matrix given by (3.17) can be written as

$$(I + E^T C^{-1} E)^{-1} = \begin{bmatrix} \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N+1} \end{bmatrix} \quad (3.18)$$

Premultiplying this matrix by E and postmultiplying by E^T gives

$$E(I + E^T C^{-1} E)^{-1} E^T =$$

$$\begin{bmatrix} \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & 0 & 0 & \frac{M}{M+N+1} \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & 0 & 0 & \frac{M}{M+N+1} \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & 0 & 0 & \frac{M}{M+N+1} \end{bmatrix} \quad (3.19)$$

and carrying out similar operations on this last matrix with matrix C^{-1} yields

$$C^{-1}E(I+E^T C^{-1}E)^{-1}E^T C^{-1} =$$

$$\begin{bmatrix} \frac{1}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & \frac{2}{M(M+N)} & \frac{2}{M(M+N)} & 0 \\ 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & 0 & \frac{4}{M(M+N+1)} \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & \frac{4}{M(M+N)} & 0 \\ 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & 0 & \frac{4}{M(M+N+1)} \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & \frac{4}{M(M+N)} & 0 \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & \frac{4}{M(M+N)} & 0 \\ 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & 0 & \frac{4}{M(M+N+1)} \end{bmatrix} \quad (3.20)$$

Finally, substituting from (3.16) and (3.20) into (3.15) results in an expression for R^{-1} as

$$R^{-1} =$$

$$\begin{bmatrix} \frac{M+N-1}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & \frac{-2}{M(M+N)} & 0 \\ 0 & \frac{2(M+N-1)}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & 0 & \frac{-4}{M(M+N+1)} \\ \frac{-2}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & \frac{-4}{M(M+N)} & 0 \\ 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{2(M+N-1)}{M(M+N+1)} & 0 & 0 & \frac{-4}{M(M+N+1)} \\ \frac{-2}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & \frac{-4}{M(M+N)} & 0 \\ \frac{-2}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & \frac{2(M+N-2)}{M(M+N)} & 0 \\ 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & 0 & \frac{2(M+N-1)}{M(M+N+1)} \end{bmatrix} \quad (3.21)$$

Note that the elements of the matrix R^{-1} simply repeat as $-4/(M(M+N+1))$ and 0 for odd numbered rows, as 0 and $-4/(M(M+N))$ for even numbered rows, and as $2(M+N-1)/(M(M+N+1))$ and $2(M+N-2)/(M(M+N))$ in the diagonal except the elements in the 0th row and the 0th column in which they follow a different pattern.

Thus, the computation of the inverse of matrix R becomes very simple. There are only six different values required to be computed, irrespective of the length (the order N) of

* The dashed-line enclosures are drawn to better visualize the repetitive nature of elements in the various segments of the matrix.

the filter. This is one of the reasons for a very slow increase in the filter design time as the order of the filter to be designed increases.

Case (ii): $N < M$ and N is even

When N is even, (3.14) and (3.16) yield

$$I + E^T C^{-1} E = \begin{bmatrix} \frac{M+N+1}{M} & 0 \\ 0 & \frac{M+N}{M} \end{bmatrix} \quad (3.22)$$

and

$$(I + E^T C^{-1} E)^{-1} = \begin{bmatrix} \frac{M}{M+N+1} & 0 \\ 0 & \frac{M}{M+N} \end{bmatrix} \quad (3.23)$$

The result of pre and post multiplication of this last matrix by E and E^T , respectively, becomes

$$E(I + E^T C^{-1} E)^{-1} E^T =$$

$$\begin{bmatrix} \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} & 0 & \frac{M}{M+N+1} \end{bmatrix} \quad (3.24)$$

As in the case of odd N , the matrix given by (3.24) is pre and post multiplied by C^{-1} , giving

$$C^{-1}E(I+E^T C^{-1}E)^{-1}E^T C^{-1} =$$

$$\begin{bmatrix} \frac{1}{M(M+N+1)} & 0 & \frac{2}{M(M+N+1)} & 0 & \frac{2}{M(M+N+1)} & 0 & \frac{2}{M(M+N+1)} \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 \\ \frac{2}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 \\ \frac{2}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 \\ \frac{2}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} & 0 & \frac{4}{M(M+N+1)} \end{bmatrix} \quad (3.25)$$

Substituting from (3.16) and (3.25) into (3.15) yields a closed form for R^{-1} expressed as

$$R^{-1} =$$

$$\begin{bmatrix} \frac{M+N}{M(M+N+1)} & 0 & \frac{-2}{M(M+N+1)} & 0 & \frac{-2}{M(M+N+1)} & 0 & \frac{-2}{M(M+N+1)} \\ 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{0}{M(M+N)} & 0 & \frac{0}{M(M+N)} & 0 \\ \frac{-2}{M(M+N+1)} & 0 & \frac{2(M+N-1)}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} \\ 0 & \frac{-4}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 \\ \frac{-2}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{2(M+N-1)}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} \\ 0 & \frac{-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 \\ \frac{-2}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{2(M+N-1)}{M(M+N+1)} \end{bmatrix} \quad (3.26)$$

Case (iii): $N=M$ and N is odd

In this case the equation corresponding to (3.18) becomes

$$(I+E^T C^{-1}E)^{-1} = \begin{bmatrix} \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N} \end{bmatrix} \quad (3.27)$$

Pre and post multiplying the above matrix by E and E^T , respectively, yields

$$E(I+E^T C^{-1}E)^{-1}E^T =$$

$$\begin{bmatrix} \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & 0 & \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & \frac{M}{M+N} & 0 & \frac{M}{M+N} \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & 0 & \frac{M}{M+N} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & \frac{M}{M+N} & 0 & \frac{M}{M+N} \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & 0 & \frac{M}{M+N} & 0 \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & \frac{M}{M+N} & 0 & \frac{M}{M+N} \end{bmatrix}$$

(3.28)

The result of pre and post multiplying the above matrix by C^{-1} is given by

$$C^{-1}E(I+E^T C^{-1}E)^{-1}E^T C^{-1} =$$

$$\begin{bmatrix} \frac{1}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & 0 & \frac{2}{M(M+N)} & 0 \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & \frac{4}{M(M+N)} & 0 & \frac{2}{M(M+N)} \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & 0 & \frac{4}{M(M+N)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & \frac{4}{M(M+N)} & 0 & \frac{2}{M(M+N)} \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & 0 & \frac{4}{M(M+N)} & 0 \\ 0 & \frac{2}{M(M+N)} & 0 & \frac{2}{M(M+N)} & \frac{2}{M(M+N)} & 0 & \frac{1}{M(M+N)} \end{bmatrix}$$

(3.29)

Subtracting the matrix given by (3.29) from C^{-1} results in R^{-1} as

$$R^{-1} =$$

$$\begin{bmatrix} \frac{M+N-1}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & 0 & \frac{-2}{M(M+N)} & 0 \\ 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & \frac{-4}{M(M+N)} & 0 & \frac{-2}{M(M+N)} \\ \frac{-2}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & 0 & \frac{-4}{M(M+N)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-2}{M(M+N)} \\ \frac{-2}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 \\ 0 & \frac{-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & \frac{-2}{M(M+N)} & 0 & \frac{M+N-1}{M(M+N)} \end{bmatrix}$$

(3.30)

Since $N=M$, (3.30) can further be simplified as

$$R^{-1} =$$

$$\begin{bmatrix} \frac{2N+1}{-2N^2} & 0 & \frac{-1}{N^2} & 0 & 0 & \frac{-1}{N^2} & 0 \\ 0 & \frac{2(N-1)}{N^2} & 0 & \frac{-2}{N^2} & \frac{-2}{N^2} & 0 & \frac{-1}{N^2} \\ \frac{-1}{N^2} & 0 & \frac{2(N-1)}{N^2} & 0 & 0 & \frac{-2}{N^2} & 0 \\ 0 & \frac{-2}{N^2} & 0 & \frac{-2}{N^2} & \frac{2(N-1)}{N^2} & 0 & \frac{-1}{N^2} \\ \frac{-1}{N^2} & 0 & \frac{-2}{N^2} & 0 & 0 & \frac{2(N-1)}{N^2} & 0 \\ 0 & \frac{-1}{N^2} & 0 & \frac{-1}{N^2} & \frac{-1}{N^2} & 0 & \frac{2N-1}{N^2} \end{bmatrix} \quad (3.31)$$

Note that, similar to (3.21), the computation of matrix R^{-1} of (3.31) requires the evaluation of only four different elements.

Case (iv): $N = M$ and N is even

In this case the results of $(I+E^T C^{-1}E)^{-1}$ will be the same as given in (3.27). However, the results corresponding to (3.28), (3.29), (3.30) and (3.31) take the following forms.

$$E(I+E^T C^{-1}E)^{-1}E^T =$$

$$\begin{bmatrix} \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} \\ 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 \\ \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} & 0 & \frac{M}{M+N} \end{bmatrix} \quad (3.32)$$

$$C^{-1}E(I+E^T C^{-1}E)^{-1}E^T C^{-1} =$$

$$\begin{bmatrix} \frac{1}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & \frac{1}{M(M+N)} \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{2}{M(M+N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{2}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{2}{M(M+N)} \\ 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 & \frac{4}{M(M+N)} & 0 \\ \frac{1}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & \frac{2}{M(M+N)} & 0 & \frac{1}{M(M+N)} \end{bmatrix} \quad (3.33)$$

$$R^{-1} =$$

$$\begin{bmatrix} \frac{M+N-1}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{-1}{M(M+N)} \\ 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 \\ \frac{-2}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{-2}{M(M+N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-2}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 & \frac{-2}{M(M+N)} \\ 0 & \frac{-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 & \frac{2(M+N-2)}{M(M+N)} & 0 \\ \frac{-1}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{M+N-1}{M(M+N)} \end{bmatrix} \quad (3.34)$$

The matrix R^{-1} given by (3.34) can further be simplified as

$$R^{-1} =$$

$$\begin{bmatrix} \frac{2N-1}{2N^2} & 0 & \frac{-1}{N^2} & 0 & \frac{-1}{N^2} & 0 & \frac{-1}{2N^2} \\ 0 & \frac{2(N-1)}{N^2} & 0 & \frac{-2}{N^2} & 0 & \frac{-2}{N^2} & 0 \\ \frac{-1}{N^2} & 0 & \frac{2(N-1)}{N^2} & 0 & \frac{-2}{N^2} & 0 & \frac{-1}{N^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{N^2} & 0 & \frac{-2}{N^2} & 0 & \frac{2(N-1)}{N^2} & 0 & \frac{-1}{N^2} \\ 0 & \frac{-2}{N^2} & 0 & \frac{-2}{N^2} & 0 & \frac{2(N-1)}{N^2} & 0 \\ \frac{-1}{2N^2} & 0 & \frac{-1}{N^2} & 0 & \frac{-1}{N^2} & 0 & \frac{2N-1}{N^2} \end{bmatrix} \quad (3.35)$$

3.2.4 The Coefficient Matrix A

If a matrix S is defined as

$$S = R^{-1}P^T \quad (3.36)$$

then (2.25) can be rewritten in the form

$$A = SH_S S^T \quad (3.37)$$

In view of (3.37), the $(N+1) \times (M+1)$ matrix S will be called *inverse frequency response transform matrix* (IFRTM). The specified frequency response matrix is transformed into the coefficient matrix, when it is operated on by the S matrix. Even for the case $M=N$, the coefficient matrix A should be evaluated using (3.37) as it will require less computations than when (2.20) is used.

Once the matrix R^{-1} has been evaluated, one can compute the matrix S using (3.36). Finally, with the knowledge of the specified frequency response, the coefficient matrix A is obtained from (3.37). After some simple manipulation using (B.1), the elements of the S matrix can be directly obtained from the knowledge of M and N . Since the formats of R^{-1} and P^T matrices vary for even and odd values of N and also for $N < M$ and $N = M$, there are four different sets of values for the elements of the S matrix.

Consider the case when $N < M$, and N is odd. Substituting from (2.17) and (3.21) into (3.36), we have

$$S = R^{-1}P^T =$$

$$\begin{bmatrix} \frac{(M+N+1)-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 & \frac{-2}{M(M+N)} & 0 \\ 0 & \frac{2(M+N+1)-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} \\ \frac{-4}{M(M+N)} & 0 & \frac{2(M+N)-4}{M(M+N)} & 0 & \frac{-4}{M(M+N)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{-4}{M(M+N+1)} & 0 & \frac{2(M+N+1)-4}{M(M+N+1)} \end{bmatrix} \times$$

$$\begin{bmatrix} \cos(0 \times 0 \times \frac{\pi}{M}) & \cos(0 \times 1 \times \frac{\pi}{M}) & \cos(0 \times 2 \times \frac{\pi}{M}) & \dots & \cos(0 \times l \times \frac{\pi}{M}) & \dots & \cos(0 \times M \times \frac{\pi}{M}) \\ \cos(1 \times 0 \times \frac{\pi}{M}) & \cos(1 \times 1 \times \frac{\pi}{M}) & \cos(1 \times 2 \times \frac{\pi}{M}) & \dots & \cos(1 \times l \times \frac{\pi}{M}) & \dots & \cos(1 \times M \times \frac{\pi}{M}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos(k \times 0 \times \frac{\pi}{M}) & \cos(k \times 1 \times \frac{\pi}{M}) & \cos(k \times 2 \times \frac{\pi}{M}) & \dots & \cos(k \times l \times \frac{\pi}{M}) & \dots & \cos(k \times M \times \frac{\pi}{M}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cos(N \times 0 \times \frac{\pi}{M}) & \cos(N \times 1 \times \frac{\pi}{M}) & \cos(N \times 2 \times \frac{\pi}{M}) & \dots & \cos(N \times l \times \frac{\pi}{M}) & \dots & \cos(N \times M \times \frac{\pi}{M}) \end{bmatrix} \quad (3.38)$$

As noted in Sec. 3.2.3, matrices R^{-1} and P have certain regularity in their structure. Thus, the determination of the element $S(i, l)$ of the S matrix, using (3.38), can be simplified by study of the following seven cases.

Case (i): $i = 0$ and $l = 0$ or $l = M$

$$S(i, l) = \frac{M+N+1}{M(M+N)} \frac{2}{M(M+N)} \cdot \frac{1}{2} \sum_{k=0}^N 1 = \frac{1}{M+N} \quad (3.39)$$

Case (ii): i is even and $l = 0$ or $l = M$

$$S(i, l) = \frac{2}{M(M+N)} + \frac{2(M+N)}{M(M+N)} \frac{4}{M(M+N)} \cdot \frac{1}{2} \sum_{k=0}^N 1 = \frac{2}{M+N} \quad (3.40)$$

Case (iii): i is odd and $l = 0$

$$S(i, l) = \frac{2(M+N+1)}{M(M+N+1)} \frac{4}{M(M+N+1)} \cdot \frac{1}{2} \sum_{k=0}^N 1 = \frac{2}{M+N+1} \quad (3.41)$$

Case (iv): i is odd and $l = M$

$$\begin{aligned} S(i, l) &= \frac{-2(M+N+1)}{M(M+N+1)} \cdot \frac{4}{M(M+N+1)} \cdot \frac{1}{2} \sum_{k=0}^N -1 \\ &= \frac{-2}{M+N+1} \end{aligned} \quad (3.42)$$

Case (v): $i = 0$ and $0 < l < M$

$$S(i, l) = \frac{(M+N+1)}{M(M+N)} \cdot \frac{2}{M(M+N)} \sum_{k=0}^{(N-1)/2} \cos(2kl \frac{\pi}{M}) \quad (3.43)$$

Using (B.1), the second term on the right side of (3.43) can be written as

$$\sum_{k=0}^{(N-1)/2} \cos(2kl \frac{\pi}{M}) = \frac{\cos[l(N-1)\frac{\pi}{2M}] \sin[l(N+1)\frac{\pi}{2M}]}{\sin(l\frac{\pi}{M})} \quad (3.44)$$

Thus, substituting from (3.44) into (3.43) yields

$$S(i, l) = \frac{1}{M(M+N)} \left[(M+N+1) - \frac{2 \cos[l(N-1)\frac{\pi}{2M}] \sin[l(N+1)\frac{\pi}{2M}]}{\sin(l\frac{\pi}{M})} \right] \quad (3.45)$$

Case (vi): i is even and $0 < l < M$

$$S(i, l) = \frac{2}{M(M+N)} + \frac{2(M+N)}{M(M+N)} \cos(il \frac{\pi}{M}) - \frac{4}{M(M+N)} \sum_{k=0}^{(N-1)/2} \cos(2kl \frac{\pi}{M}) \quad (3.46)$$

Using (3.44), (3.46) can be rewritten as

$$\begin{aligned} S(i, l) &= \frac{2}{M(M+N)} + \frac{2}{M} \cos(li \frac{\pi}{M}) \\ &\quad - \frac{4}{M(M+N)} \frac{\cos[l(N-1)\frac{\pi}{2M}] \sin[l(N+1)\frac{\pi}{2M}]}{\sin(l\frac{\pi}{M})} \end{aligned} \quad (3.47)$$

Case (vii): i is odd and $0 < l < M$

$$S(i, l) = \frac{2(M+N+1)}{M(M+N+1)} \cos\left(li \frac{\pi}{M}\right) - \frac{4}{M(M+N+1)} \left[\sum_{k=0}^N \cos\left(kl \frac{\pi}{M}\right) - \sum_{k=0}^{(N-1)/2} \cos\left(2kl \frac{\pi}{M}\right) \right] \quad (3.48)$$

Using (B.2), (3.48) can be expressed as

$$S(i, l) = \frac{2}{M} \cos\left(li \frac{\pi}{M}\right) - \frac{4}{M(M+N+1)} \left[\frac{\cos\left[l(N-1) \frac{\pi}{2M}\right] \sin\left[l(N+1) \frac{\pi}{2M}\right]}{\sin\left(l \frac{\pi}{M}\right)} - \frac{\cos\left(lN \frac{\pi}{2M}\right) \sin\left[l(N+1) \frac{\pi}{2M}\right]}{\sin\left(l \frac{\pi}{2M}\right)} \right] \quad (3.49)$$

It should be noted that (3.47) and (3.49) are quite general and they can be used along with the L'Hopital rule for cases (i), (ii), (iii) and (iv) as well. However, (3.47) and (3.49) have not been used for these cases, since their treatment is simpler without using these equations.

Following a treatment as carried out above, expressions for $S(i, l)$ can be derived for the $N < M$ and even N , and $N = M$ cases. The expressions for the elements of matrix S when $N < M$ are given in Table 3.1, where the expressions for the functions used in this table can be obtained from Table 3.2. Likewise, for the $N = M$ case, the expressions for the matrix elements can be obtained from Tables 3.3 and 3.4.

Once the elements of the S matrix have been evaluated using Tables 3.1 and 3.2 or Tables 3.3 and 3.4, (3.37) can be used to compute the A matrix. Alternatively, the elements of A can be obtained by evaluating the following summation.

$$a(i, j) = \sum_{k=0}^M \sum_{l=0}^M S(i, l) H_S(l, k) S(j, k) \quad (3.50)$$

TABLE 3.1

Elements of the Inverse Frequency Response Transform Matrix S for $N < M$

| i | l | $S(i, l)$ |
|------|-------------|---|
| 0 | 0 or M | d_1 |
| even | 0 or M | $2d_1$ |
| odd | 0 | e_1 |
| odd | M | $-e_1$ |
| 0 | $0 < l < M$ | $[d_1 + f_1(i, l) - d_1 \times f_{e1}(l)]/M$ |
| even | $0 < l < M$ | $2[d_1 + f_1(i, l) - d_1 \times f_{e1}(l)]/M$ |
| odd | $0 < l < M$ | $2[f_1(i, l) + e_1 \times f_{o1}(l)]/M$ |

TABLE 3.2

Expressions for the Functions d_1 , e_1 , f_{e1} , f_{o1} and f_1

| FUNCTION \ N | EVEN | ODD |
|--------------|--|--|
| d_1 | $1/(M+N+1)$ | $1/(M+N)$ |
| e_1 | $2/(M+N)$ | $2/(M+N+1)$ |
| $f_{e1}(l)$ | $2 \cos(lN\pi/2M) \sin[l(N+2)\pi/2M] / \sin(l\pi/M)$ | $2 \cos[l(N-1)\pi/2M] \sin[l(N+1)\pi/2M] / \sin(l\pi/M)$ |
| $f_{o1}(l)$ | $\cos(lN\pi/2M) \sin[l(N+2)\pi/2M] / \sin(l\pi/M) - \cos(lN\pi/2M) \sin[l(N+1)\pi/2M] / \sin(l\pi/2M)$ | $\cos[l(N-1)\pi/2M] \sin[l(N+1)\pi/2M] / \sin(l\pi/M) - \cos(lN\pi/2M) \sin[l(N+1)\pi/2M] / \sin(l\pi/2M)$ |
| $f_1(l, l)$ | $\cos(l\pi/M)$ | $\cos(l\pi/M)$ |

TABLE 3.3

Elements of the Inverse Frequency Response Transform Matrix S for $N = M$

| i | l | $S(i, l)$ |
|----------------------|-------------|--------------------------------------|
| 0 | 0 or N | $d_2/2$ |
| N | 0 | $d_2/2$ |
| $0 < i < N$ | 0 | d_2 |
| even, $2 \leq i < N$ | N | d_2 |
| odd, $1 \leq i < N$ | N | $-d_2$ |
| N | N | e_2 |
| 0 | $0 < l < N$ | $[d_2^2 + 2f_2(i, l) + f_{e2}(l)]/2$ |
| N (even) | $0 < l < N$ | $[d_2^2 + 2f_2(i, l) + f_{e2}(l)]/2$ |
| N (odd) | $0 < l < N$ | $f_2(i, l) + f_{o2}(l)/2$ |
| even, $2 \leq i < N$ | $0 < l < N$ | $d_2^2 + 2f_2(i, l) + f_{e2}(l)$ |
| odd, $1 \leq i < N$ | $0 < l < N$ | $2f_2(i, l) + f_{o2}(l)$ |

TABLE 3.1

Expressions for the Functions d_2 , e_2 , f_{e2} , f_{o2} , and f_2

| FUNCTION | EVEN | ODD |
|-------------|--|---|
| d_2 | $1/N$ | $1/N$ |
| e_2 | $1/2N$ | $-1/2N$ |
| $f_{e2}(l)$ | $\{(-1)^l - 2\cos(l\pi/2)\sin[l(N+2)\pi/2N]/\sin(l\pi/N)\}/N^2$ | $-2\{\cos[l(N-1)\pi/2N]\sin[l(N+1)\pi/2N]/\sin(l\pi/N)\}/N^2$ |
| $f_{o2}(l)$ | $\{2\cos(l\pi/2)\sin[l(N+2)\pi/2N]/\sin(l\pi/N) - 2\cos(l\pi/2)\sin[l(N+1)\pi/2N]/\sin(l\pi/2N)\}/N^2$ | $\{(-1)^l - 2\cos(l\pi/2)\sin[l(N+1)\pi/2N]/\sin(l\pi/2N) + 2\cos[l(N-1)\pi/2N]\sin[l(N+1)\pi/2N]/\sin(l\pi/N)\}/N^2$ |
| $f_2(l, l)$ | $[\cos(l\pi/N)]/N$ | $[\cos(l\pi/N)]/N$ |

3.3 QUADRANTALLY ANTISYMMETRIC FILTERS

In this section, the design of FIR filters possessing quadrantly antisymmetric frequency response is considered. The study of the properties of matrices Q and U leads to even simpler formulae for the design of such filters.

3.3.1 Properties of the Frequency Response Transform Matrix Q and Matrix U

In Secs. 3.2.1 and 3.2.2, the properties of matrices P and R were discussed, then these properties were used in obtaining very simple formats for R^{-1} and S . Similar properties can also be utilized in the design of quadrantly antisymmetric FIR filters. The transform matrix Q possesses the same properties as P . The elements of this matrix, as specified in (2.29), are ordered structure. There are only M different elements in the entire matrix and a different subset of these elements appears in each column. The value of an element of this matrix is $\sin(i\pi/M)$, where depending on the position of the element in the matrix, i assumes a value in $\{1, 2, \dots, M\}$. Specifically, the value of the element in the n th row and k th column is given by $\sin(n \times k \times \pi/M)$.

The expressions for the elements of U can be obtained by following a method similar to the one used for obtaining R . The element $U(n_1, n_2)$ ($n_1 = 1, \dots, N$; $n_2 = 1, \dots, N$) can be obtained by using the trigonometric relation given by (B.1) and carrying out some simple manipulations as in the following three cases.

Case (i): $n_1 = n_2$

In this case, (2.32) can be written as

$$U(n_1, n_2) = \sum_{k=1}^M \sin^2(n_1 k \frac{\pi}{M}) = \sum_{k=0}^M \sin^2(n_1 k \frac{\pi}{M}) \quad (3.61)$$

In order to use (B.1), (3.51) can be rewritten as

$$\begin{aligned} U(n_1, n_2) &= \sum_{k=0}^M \left[\frac{1 - \cos(2n_1 k \frac{\pi}{M})}{2} \right] \\ &= \frac{M}{2} + \frac{1}{2} - \frac{1}{2} \sum_{k=0}^M \cos(2n_1 k \frac{\pi}{M}) \end{aligned} \quad (3.52)$$

Now, by using (B.1) it can be shown that the value of the summation given by the third term of (3.52) is $1/2$. Thus,

$$U(n_1, n_2) = \frac{M}{2}$$

Case (ii): $n_1 \neq n_2$ and $(n_1 + n_2)$ is even

From (2.32),

$$\begin{aligned} U(n_1, n_2) &= \frac{1}{2} \left[\sum_{k=1}^M \cos(ak \frac{\pi}{M}) - \sum_{k=1}^M \cos(bk \frac{\pi}{M}) \right] \\ &= \frac{1}{2} \left[\sum_{k=0}^M \cos(ak \frac{\pi}{M}) - \sum_{k=0}^M \cos(bk \frac{\pi}{M}) \right] \end{aligned} \quad (3.53)$$

where $a = (n_1 + n_2)$ and $b = (n_1 - n_2)$ are even. Using (B.1), it can be found that the first and the second summations of the right side of (3.53) are equal. Therefore,

$$U(n_1, n_2) = 0$$

Case (iii): $(n_1 + n_2)$ is odd

$U(n_1, n_2)$ can be expressed as in (3.53) where $a = (n_1 + n_2)$ and $b = (n_1 - n_2)$, in this case, are odd. The two summations of the right side of (3.53) are both zero. Thus,

$$U(n_1, n_2) = 0$$

The above analysis leads to a greatly simplified matrix, U , which now takes the form

$$U = \begin{bmatrix} \frac{M}{2} & \bigcirc \\ \frac{M}{2} & \bigcirc \\ \bigcirc & \frac{M}{2} \end{bmatrix} \quad (3.54)$$

In the above analysis, N is assumed to be less than M . Unlike the case of the quadrantly symmetric response, N cannot be chosen to be equal to M , since $U(N, N)$ would then become 0, resulting in U becoming a singular matrix. The inverse of the U matrix given by (3.54) can be expressed simply as

$$U^{-1} = \begin{bmatrix} \frac{2}{M} & \bigcirc \\ \frac{2}{M} & \bigcirc \\ \bigcirc & \frac{2}{M} \end{bmatrix} \quad (3.55)$$

3.3.2 The Coefficient Matrix B

As in the case of symmetric response, here too, an inverse frequency response transform matrix V is defined and it is given by

$$V = U^{-1}Q^T \quad (3.56)$$

Thus, (2.30) can be rewritten in the form

$$B = VH_A V^T \quad (3.57)$$

Since U^{-1} is diagonal matrix, it can be easily found that $V(i, l) = 2Q(i, l)/M$. Then, (3.57) can be used to compute the B matrix. However, the elements of B can be obtained much more simply by noting that

$$b(i,j) = \sum_{k=1}^M \sum_{l=1}^M V(i,l) H_A(l,k) V(j,k)$$

In this equation, replacing $V(i,l)$ by $2Q(i,l)/M$ yields

$$b(i,j) = \sum_{k=1}^M \sum_{l=1}^M \frac{4}{M^2} Q(i,l) H_A(l,k) Q(j,k) \quad (3.58)$$

Finally, by using the expression of $Q(i,l)$ from (2.29), (3.58) becomes

$$b(i,j) = \frac{4}{M^2} \sum_{k=1}^M \sum_{l=1}^M \sin(il\frac{\pi}{M}) H_A(l,k) \sin(jk\frac{\pi}{M}) \quad (3.59)$$

3.4 CONCLUSION

The least square solution to the design problem of FIR filters has resulted in the development of a number of matrices. Properties of these matrices have been studied from the point of view of reducing the computational complexity of the filter design. It has been shown that no optimization technique need to be applied. Simple expressions for the elements of the inverse frequency response transform matrices have been obtained. The specified frequency response matrix, when operated on by this matrix, gives readily the designed filter's coefficients.

CHAPTER IV

DESIGN EXAMPLES

4.1 INTRODUCTION

In this chapter, a number of design examples illustrating the technique of 2-D FIR filter design introduced in Chapters II and III and its efficiency, are presented. Results of the proposed design, in terms of CPU design time and maximum deviation in frequency response from the desired response, are compared with those obtained by using the l_p -optimization technique [27], the WLS technique employing modified Lawson's algorithm [30], and the technique using Harris' ascent algorithm [28]. Contour and perspective plots illustrating the magnitude frequency response of designed filters are presented.

4.2 EXAMPLES

Example 4.1: 90° Fan Filter

In this example an FIR fan filter approximating the magnitude response shown in Fig. 4.1, is designed. The specified frequency response $H_S(\omega_1, \omega_2)$ has zero phase and magnitude of unity in the passband (PB) and zero in the stopband (SB), and the values vary linearly between zero and one in the transition band (TB). A sampling interval of 0.05π ($M = 20$) is chosen in both dimensions. The filter is designed with 21×21 coefficients ($N = 20$). A design time of 2.1 seconds was recorded on a VAX 11/780 computer. The contour plot and the perspective plot of the magnitude-frequency response are shown in Fig. 4.2 and Fig. 4.3, respectively. It can be seen that the frequency response of the designed filter is almost the same as the specified response. An l_p -optimization technique [27] to design a 17×17 -order fan filter with the same

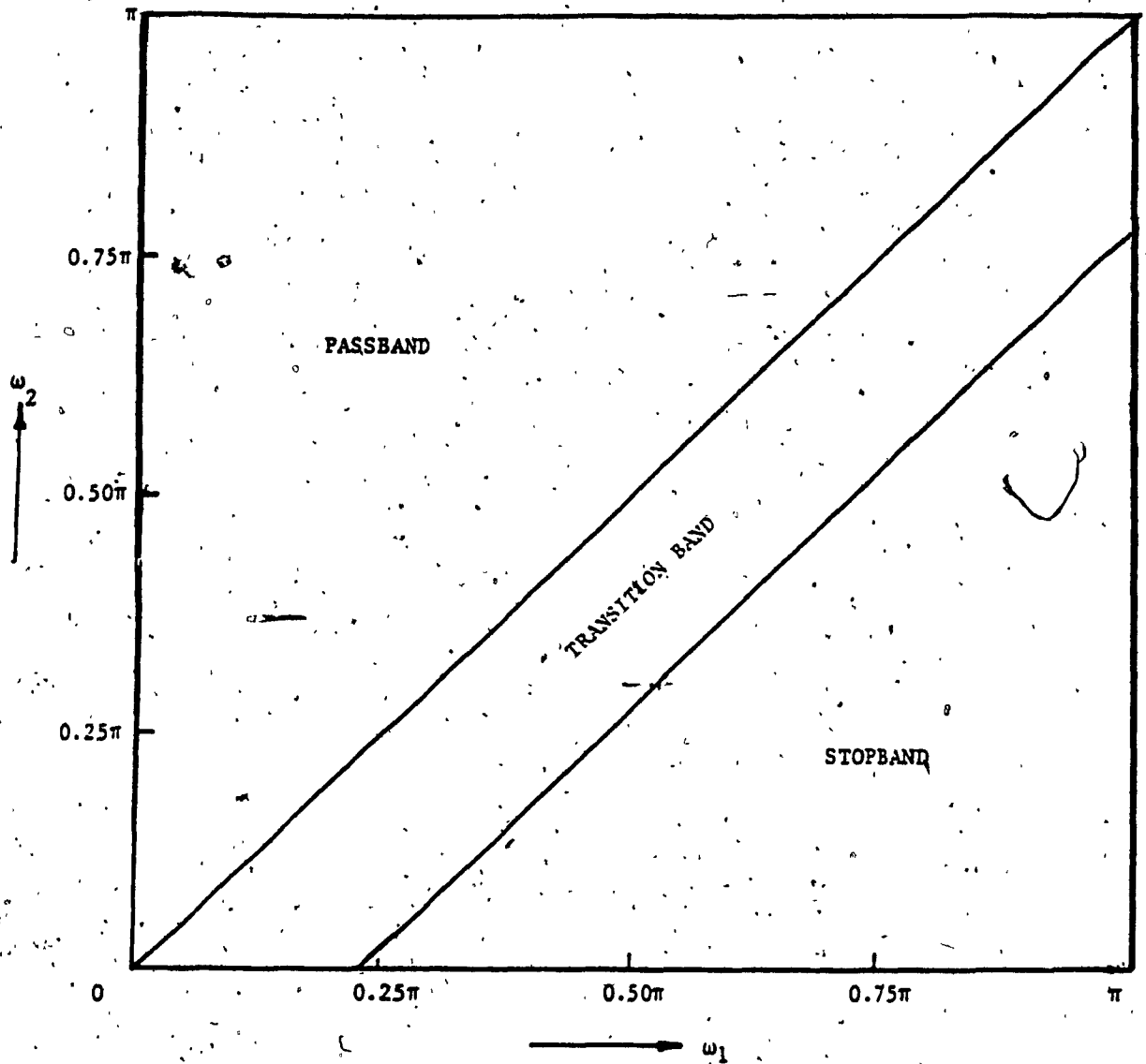


Figure 4.1 Specified Magnitude-Frequency Response of the Fan Filter of Example 4.1.

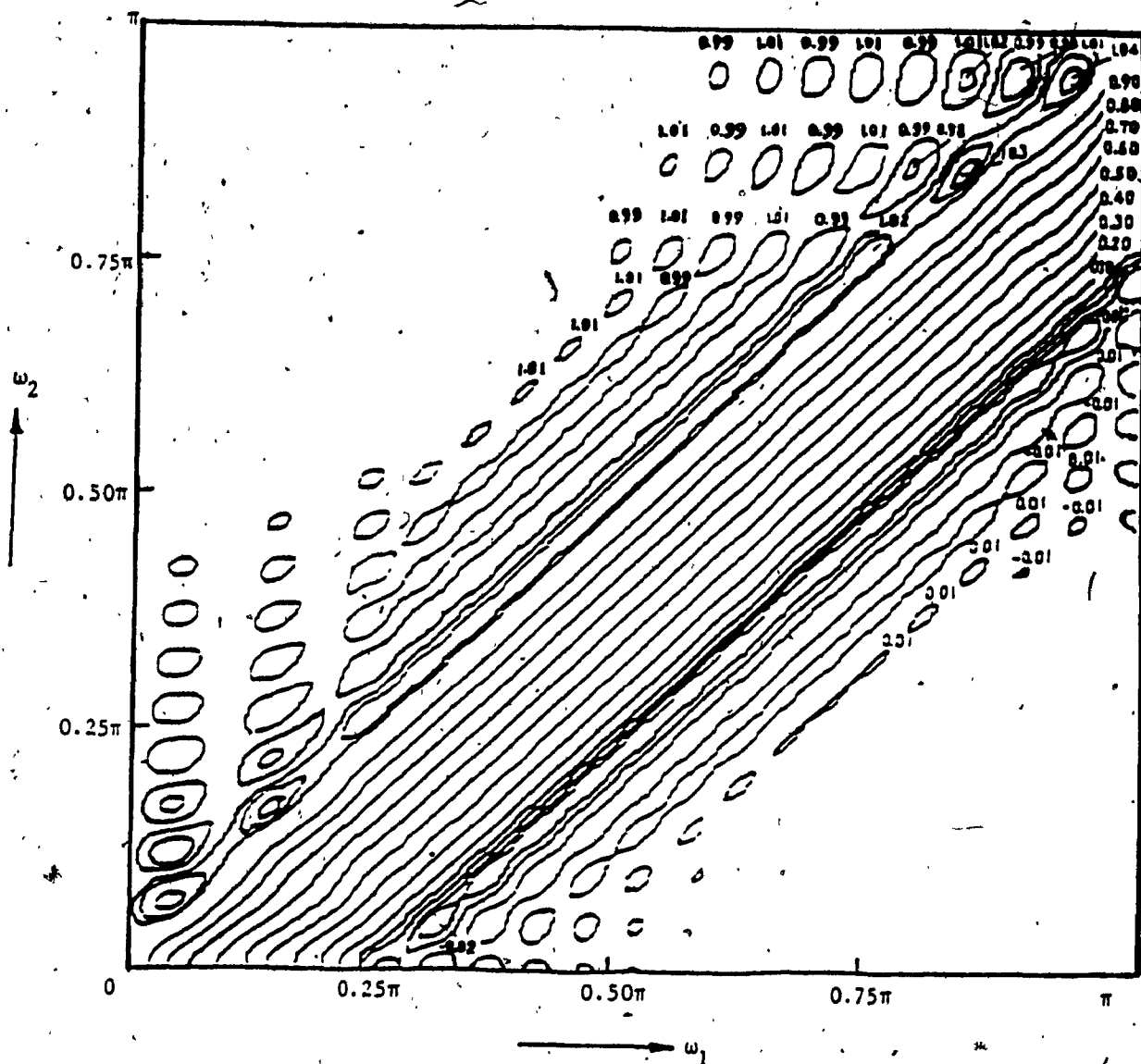


Figure 4.2 Contour Plot of the Magnitude-Frequency Response of the Designed Fan Filter of Example 4.1.

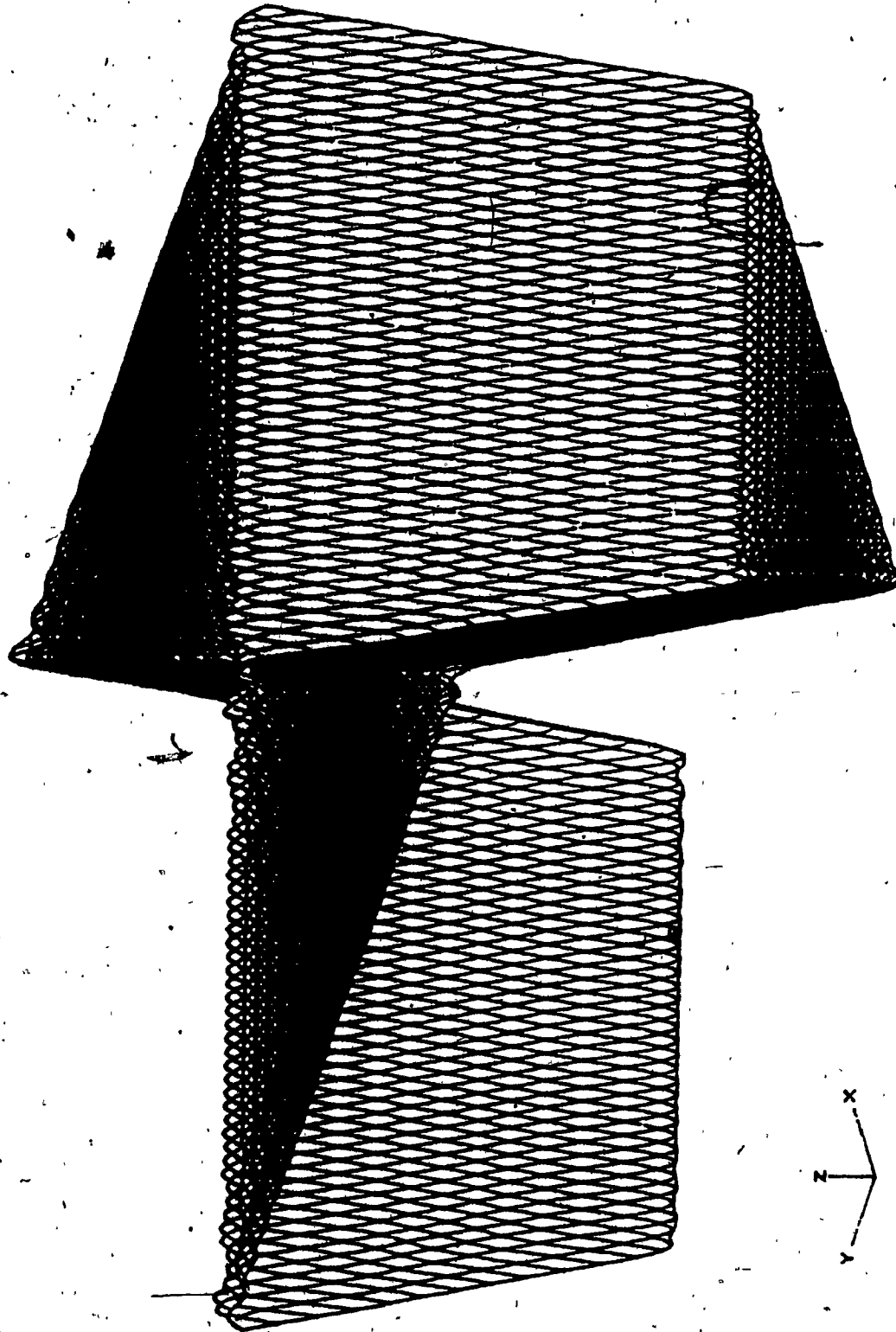


Figure 4.3 Perspective Plot of the Magnitude-Frequency Response of
the Designed Fan Filter of Example 4.1.

z:-0.022 to 1.096

specification and using the same sampling interval requires 4 minutes and 50 seconds on a Burroughs B6700 computer.

Example 4.2: Circularly-Symmetric Lowpass Filter

In this example, a circularly symmetric lowpass filter with the desired response shown in fig. 4.4 is designed. The prescribed magnitude $H_S(\omega_1, \omega_2)$ is 1 in the passband and 0 in the stopband. The filter is designed for various passband and stopband edges, orders, and sampling densities. Table 4.1 shows a comparison of the results obtained by using Harris' ascent algorithm, the modified Lawson's algorithm [30], and the proposed technique using various specifications. The table shows the design time and the maximum deviations in frequency response for filters of three different sizes. For all the cases in the example considered, the design time is less than 2 seconds. For the same filter, the design time varies from 30 seconds to 14 minutes and 14 seconds when other techniques are used. Symbols f_c and f_s have been used to represent the edges of passband and stopband frequencies, respectively. A contour plot and ~~perspective plot~~ representing the magnitude-frequency response of the $N = 5$ lowpass filter designed with $f_c = 0.175$, $f_s = 0.325$, and a sampling density of 32, are shown in Fig. 4.5 and Fig. 4.6, respectively.

Example 4.3: Centro-Symmetric Filter

In this example, a centro-symmetric filter with the desired frequency response $H(\omega_1, \omega_2)$ of Fig. 4.7 with the prescribed magnitude of 1 in the passband and 0 in the stopband, is designed. In this case, first by using the symmetrical decomposition [31], two sub-filters with quadrantal symmetry and quadrantal antisymmetry in their responses are designed. The two responses are then combined in parallel to obtain the overall frequency response, giving

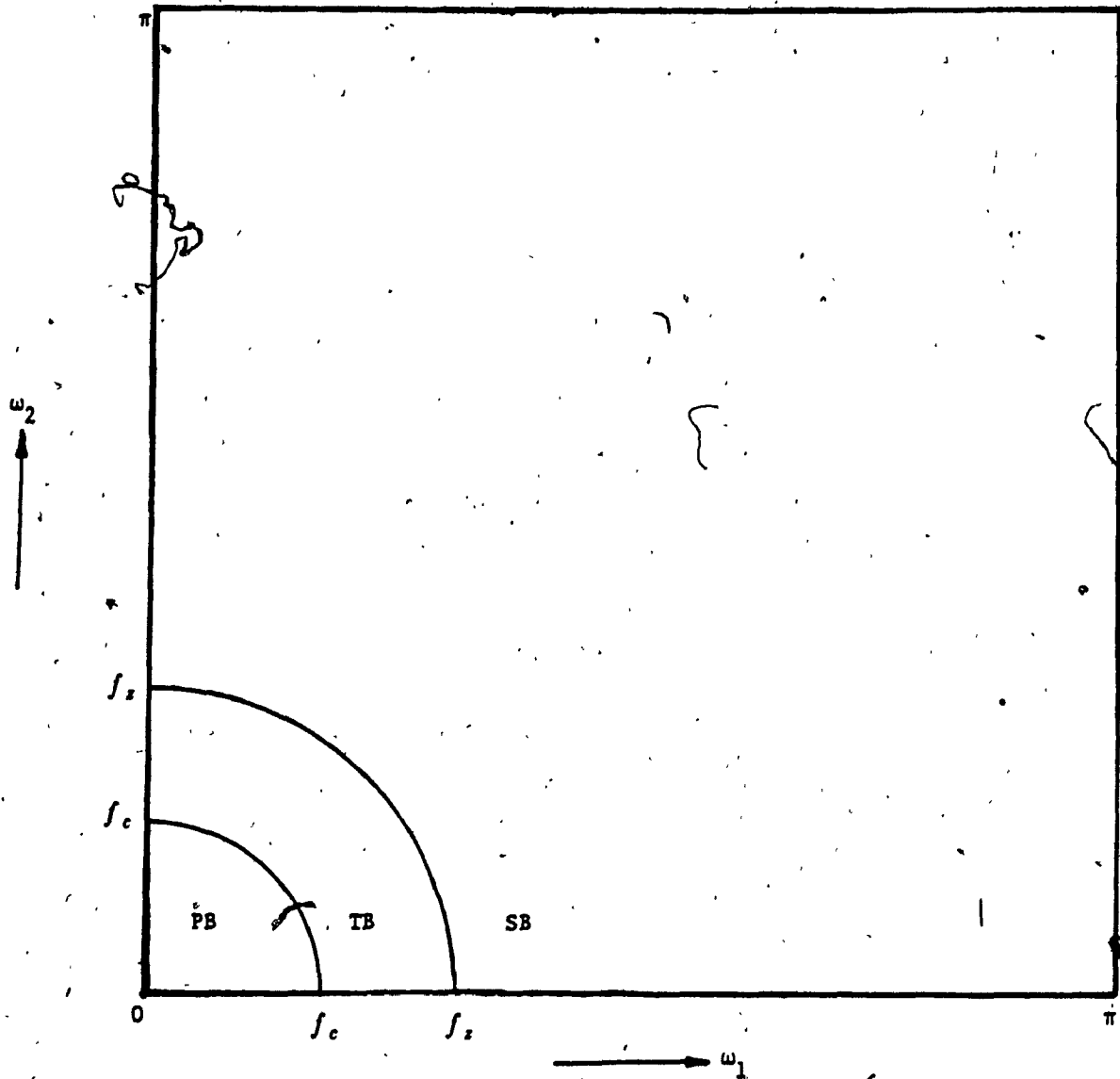


Figure 4.4 Magnitude-Frequency Response Specification of the Circularly-Symmetric Lowpass Filter of Example 4.2.

TABLE 4.1

A Comparison of Harris' Ascent Algorithm and Modified Lawson's Algorithm
with the Proposed Technique in the Design of Two-Dimensional FIR
'Circularly Symmetric Lowpass Digital Filter

| f_c | f_z | Order N | Method | Sampling Density M | Maximum Error (in dB) | Design Time* in Seconds |
|-------|-------|------------|--------------------|--------------------------|-----------------------------|----------------------------|
| 0.225 | 0.275 | 5 | Harris' Algorithm | 15 | 0.3673 (-8.7) | 53 |
| | | | Lawson's Algorithm | 32 | 0.3755 (-8.5) | 35 |
| | | | Proposed Technique | 32 | 0.3355 (-9.5) | 1.6 |
| | | 7 | Harris' Algorithm | 15 | 0.2473 (-12.1) | 183 |
| | | | Lawson's Algorithm | 21 | 0.2529 (-11.9) | 41 |
| | | | Proposed Technique | 21 | 0.2141 (-13.4) | 0.8 |
| | | 9 | Harris' Algorithm | 15 | 0.2330 (-12.6) | 496 |
| | | | Lawson's Algorithm | 16 | 0.2341 (-12.6) | 60 |
| | | | Proposed Technique | 16 | 0.2455 (-12.2) | 0.5 |
| 0.2 | 0.3 | 5 | Harris' Algorithm | 20 | 0.2670 (-11.5) | 78 |
| | | | Lawson's Algorithm | 32 | 0.2733 (-11.3) | 41 |
| | | | Proposed Technique | 32 | 0.2576 (-11.8) | 1.7 |
| | | 7 | Harris' Algorithm | 20 | 0.1272 (-17.9) | 190 |
| | | | Lawson's Algorithm | 21 | 0.1336 (-17.5) | 53 |
| | | | Proposed Technique | 21 | 0.1100 (-19.2) | 0.8 |

TABLE 4.1 (Continued)

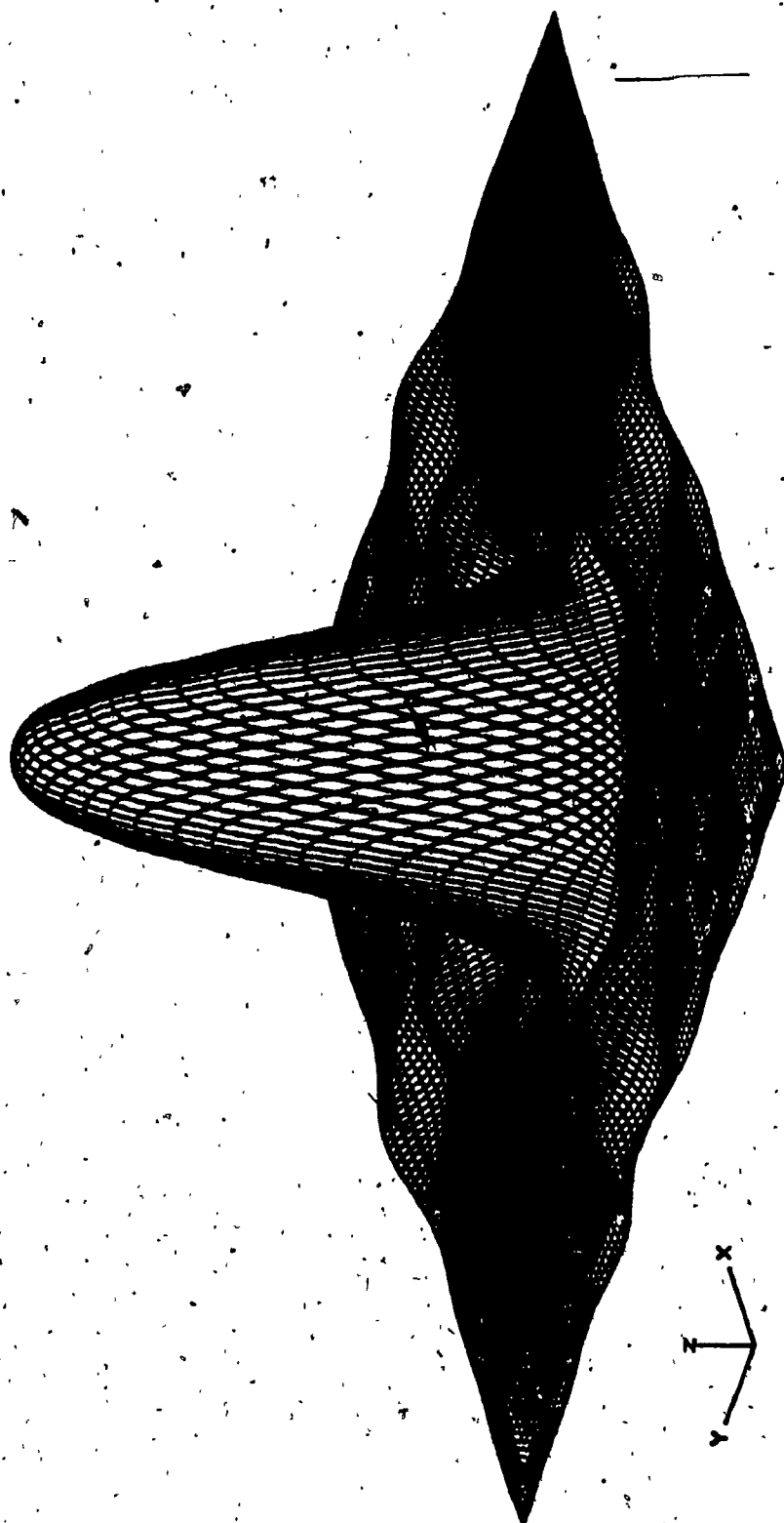
| f_c | f_z | Order N | Method | Sampling Density M | Maximum Error (in dB) | Design Time* in Seconds |
|-------|-------|------------|-----------------------|--------------------------|-----------------------------|----------------------------|
| | | 9 | Harris' Algorithm | 20 | 0.1142 (-18.8) | 854 |
| | | | Lawson's Algorithm | 16 | 0.1202 (-18.4) | 50 |
| | | | Proposed Technique | 20 | 0.1167 (-18.7) | 0.9 |
| 0.175 | 0.325 | 5 | Harris' Algorithm | 15 | 0.1905 (-14.4) | 66 |
| | | | Lawson's Algorithm | 32 | 0.1916 (-14.3) | 40 |
| | | | Proposed Algorithm | 32 | 0.1417 (-17.0) | 1.8 |
| | | 7 | Harris' Algorithm | 15 | 0.0656 (-23.7) | 102 |
| | | | Lawson's Algorithm | 21 | 0.0650 (-23.7) | 52 |
| | | | Proposed Technique | 21 | 0.0622 (-24.1) | 0.9 |
| | | 9 | Harris' Algorithm | 15 | 0.0557 (-25.0) | 491 |
| | | | Lawson's Algorithm | 16 | 0.0579 (-24.7) | 30 |
| | | | Proposed Technique | 16 | 0.0665 (-23.5) | 0.6 |

*.PDP 11/60 for Harris' algorithm and VAX 11/780 for Lawson's algorithm and the proposed technique.



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with $N = 5$, $M = 32$, $f_c = 0.175$, and $f_s = 0.325$.



21-0.090 TO 1.142

Figure 4.6 Perspective Plot of the Designed Filter of Example 4.2

with $N = 5$, $M = 32$, $f_c = 0.175$, and $f_s = 0.325$.

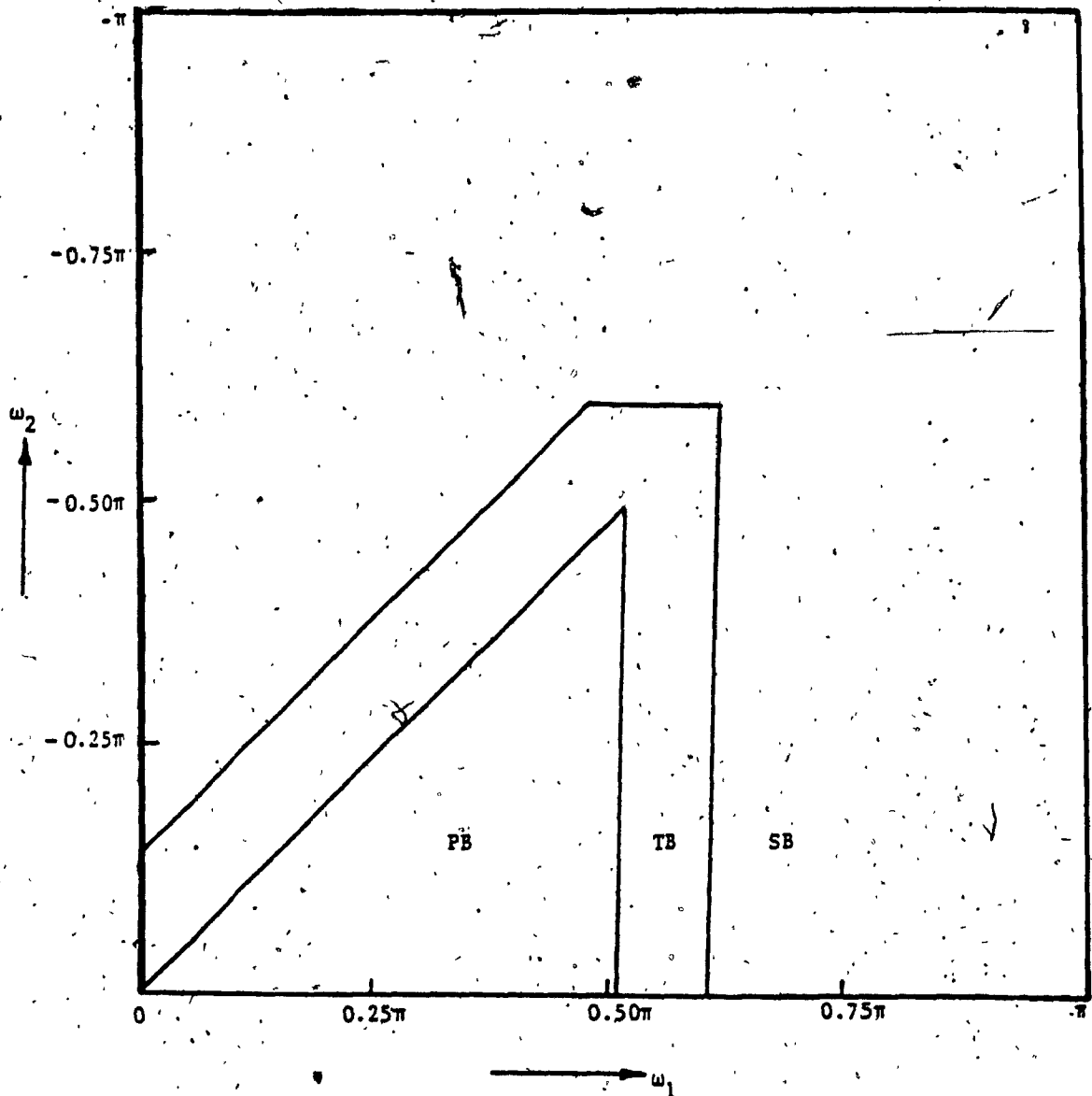


Figure 4.7 Magnitude-Frequency Response Specification for Example 4.3.

$$H(\omega_1, \omega_2) = \sum_{i=0}^N \sum_{j=0}^N a(i, j) \cos(i\omega_1) \cos(j\omega_2) + \sum_{i=1}^{N'} \sum_{j=1}^{N'} b(i, j) \sin(i\omega_1) \sin(j\omega_2)$$

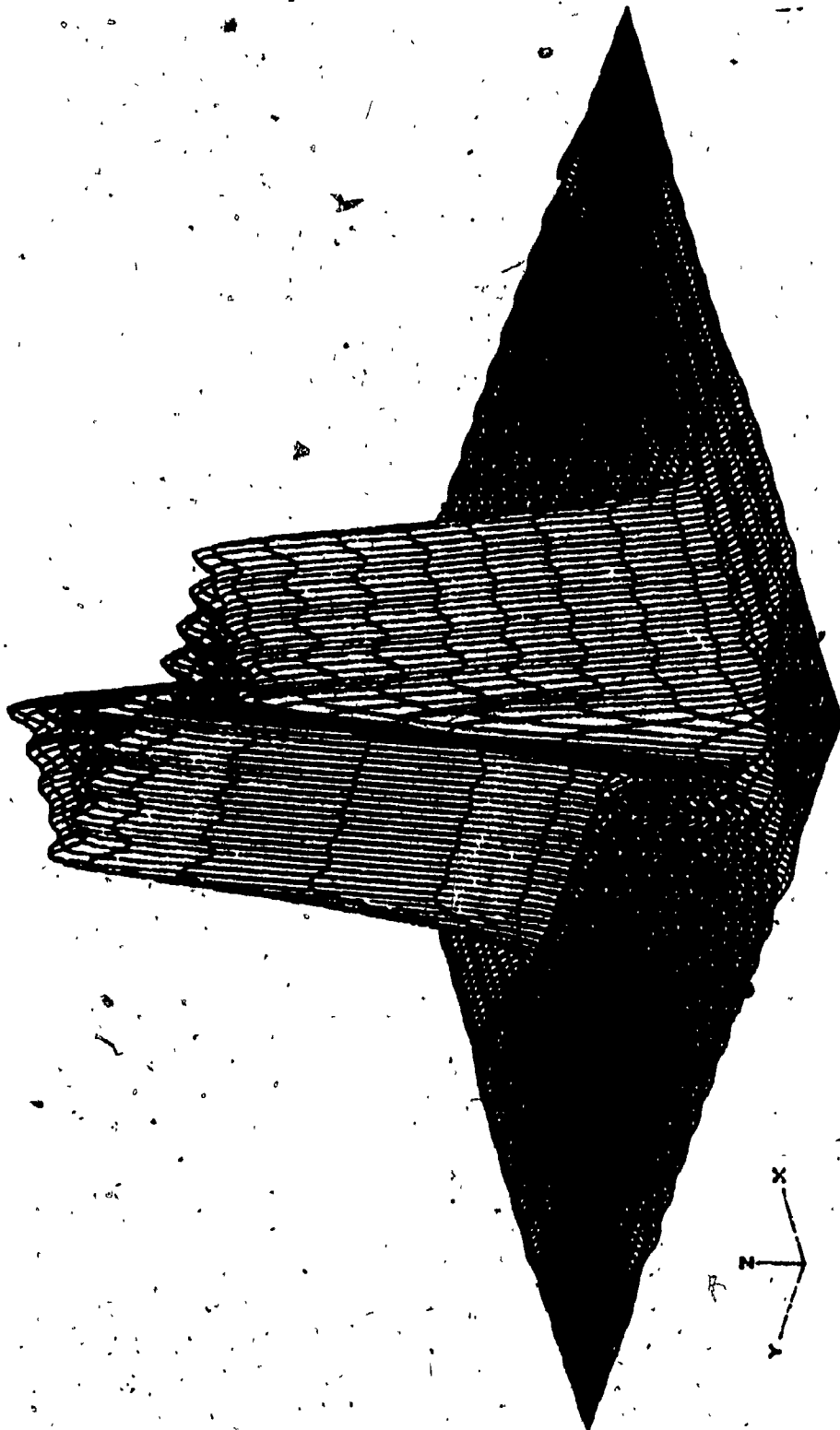
where $N \leq M$ and $N' < M$, with M being the sampling density. Thus, the specified frequency response $H(\omega_1, \omega_2)$ is decomposed into two components: (i) quadrantly symmetric component with the passband magnitude of 0.5 and the stopband magnitude of 0 and (ii) quadrantly antisymmetric component with the passband magnitude of 0.5 in the second and the fourth quadrants and -0.5 in the first and the third quadrants, respectively, and the stopband magnitude of 0. By using the proposed technique, two filters corresponding to these symmetric and antisymmetric responses are designed with 20×20 and 18×18 coefficients, respectively and with a sampling density of 20. The design time for the complete filter is 1.8 seconds. Fig. 4.8 and Fig. 4.9 show the contour and perspective plots, respectively, of the magnitude-frequency response of the complete filter. The maximum error in the passband is found to be 0.0963.

4.3 DISCUSSION AND CONCLUSION

The design examples of this chapter have demonstrated a very slow increase in the design time with the increasing order of the filter. The reason for this slow rise in design time can be found by examining the S matrix closely. The number of rows and columns of the S matrix are directly proportional to the order of filter, N , and the sampling density, M , respectively. Thus, the computational complexity is dependent on the choice of the values for M and N . Design time, however, increases very slowly as the order of the filter increases. The reason for this can be found by examining Tables 3.1 and 3.2. An increase in N would require the evaluation of functions of the type $f_1(i, l)$. Since the format of $f_1(i, l)$ is very simple, an increase in N does not lead to rapid increase in the design time. On the other hand, an increased M would require the evaluation of a large number of functions of the types $f_{e,1}(l)$, $f_{o,1}(l)$, and $f_1(i, l)$. The functions $f_{e,1}(l)$ and $f_{o,1}(l)$, however, are relatively more complex leading to a faster



1



Z: -0.080 TO 1.084

Figure 4.9 Perspective Plot of the Designed Filter in Example 4.3.

increase in the design time.

The design examples of this chapter not only have demonstrated the simplicity of the proposed technique but also have illustrated its efficiency in comparison with other techniques. From the nine examples of circularly symmetric low-pass filter designed with different specifications, sizes, and sampling frequencies (Example 4.2), it is clear that in each case, the maximum deviation in the designed filter's response from the specified one is very small and it is superior or comparable to those obtained by using other techniques. The most important point to note from the examples is that the design time using the proposed technique is significantly smaller than what it is when other techniques are used. The design time using the proposed technique ranges from a fraction of a second for low-order filters to only a few seconds for high-order filters.

CHAPTER V

CONCLUDING REMARKS

5.1 CONCLUSIONS

In this thesis an analytical least-square solution to the design problem of 2-D real zero-phase FIR filters, with quadrantly symmetric and antisymmetric frequency responses have been obtained. The proposed filter design technique allows the determination of filter's coefficients directly from its frequency response specification without employing the usual time consuming methods of optimization, iterative procedures, or matrix inversion. This analytical design solution for the class of filters has been possible because of the recognition, exposition, and exploitation of some extremely useful properties of the functions and matrices encountered in the development of the solution.

The design time is one or several orders of magnitude smaller than what it is when other techniques are used. Such a short design time, as confirmed by several illustrative examples, is due to the fact that the final design procedure essentially consists of a few matrix multiplications. Another significant advantage of the proposed design technique is that, unlike other techniques, design time is less sensitive to increase in the order of the filter. The designed filter's response is comparable or superior to those obtained by using other methods.

The proposed technique can also be applied to the design of 1-D FIR digital filters. Furthermore, it would be possible to extend the technique to the design of higher-dimensional filters.

5.2 SCOPE FOR FUTURE INVESTIGATION

The proposed technique for the design of 2-D FIR centro-symmetric filters is based on symmetrical decompositions. However, the theoretical question of whether the design based on decomposed specification and that based on undecomposed specification result in the same optimal solution or not, remains to be answered. Furthermore, it would be desirable to achieve an analytical solution to FIR design problem for any specified frequency response.

Although a great deal is known about design techniques for optimum FIR digital filters, there have not been yet established any practical design rules for such filters [39], [40]. In the design of 2-D FIR filters, it is not easily possible to establish either an approximate or exact filter order required to meet the design specifications. As a compromise, most designers resort to trial and error procedures at the expense of long design time. Since the proposed technique takes a very short design time, a designer may vary the order of the filter from its initially chosen value, step by step, until the desired precision is achieved. However, it would be attractive to find an analytic relation between the filter parameters and the error and to set up a practical set of simple design rules, for estimating the order of the filter from the desired specifications.

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APPENDIX A

SOME MATRIX RESULTS AND BIDIMENSIONAL LEAST SQUARE APPROXIMATION

Matrix Results

For any two compatible matrices **A** and **B**, the following results hold:

$$\text{tr} [\mathbf{AB}] = \text{tr} [\mathbf{BA}] \quad (\text{A.1})$$

$$\text{tr} [\mathbf{B}] = \text{tr} [\mathbf{B}^T] \quad (\text{A.2})$$

$$\text{tr} [\mathbf{A+B}] = \text{tr} [\mathbf{A}] + \text{tr} [\mathbf{B}] \quad (\text{A.3})$$

The gradient of the trace of some composite matrices comprising matrices **A**, **B**, and **C** with respect to **A** is given by

$$\frac{\partial}{\partial \mathbf{A}} [\text{tr} (\mathbf{B A C})] = \mathbf{B}^T \mathbf{C}^T \quad (\text{A.4})$$

$$\frac{\partial}{\partial \mathbf{A}} [\text{tr} (\mathbf{B A}^T \mathbf{C A B}^T)] = 2 \mathbf{C A B}^T \mathbf{B} \quad (\text{A.5})$$

Least Square Approximation

Assume that it is possible to express an $(M+1) \times (M'+1)$ frequency response matrix **H** as

$$\mathbf{H} = \mathbf{P A T}^T \quad (\text{A.6})$$

where **P** and **T** are $(M+1) \times (N+1)$ and $(M'+1) \times (N'+1)$ matrices and **A** is an $(N+1) \times (N'+1)$ coefficient matrix. The mean square error between the specified frequency response and that of the desired filter is given by

$$E = \|\hat{H} - H\|^2 = \sum_{i=0}^M \sum_{j=0}^{M'} [\hat{H}(i,j) - H(i,j)]^2 = \text{tr} [(\hat{H} - H)^T (\hat{H} - H)] \quad (\text{A.7})$$

where \hat{H} is the $(M+1) \times (M'+1)$ specified frequency response matrix. Using (A.6) into (A.7) gives

$$E = \text{tr} [(\hat{H} - \text{PAT}^T)^T (\hat{H} - \text{PAT}^T)] \quad (\text{A.8})$$

Using (A.1), (A.2), and (A.3), after some manipulations, (A.8) becomes

$$E = \text{tr} [\hat{H}^T \hat{H}] - 2 \text{tr} [\hat{H}^T \text{PAT}^T] + \text{tr} [\text{TA}^T \text{P}^T \text{PAT}^T] \quad (\text{A.9})$$

Minimization of E is obtained when

$$\frac{\partial E}{\partial A} = 0 \quad (\text{A.10})$$

The gradient of E with respect to A can be written as

$$\frac{\partial E}{\partial A} = \frac{\partial}{\partial A} (\text{tr} [\hat{H}^T \hat{H}] - 2 \text{tr} [\hat{H}^T \text{PAT}^T] + \text{tr} [\text{TA}^T \text{P}^T \text{PAT}^T]) \quad (\text{A.11})$$

Applying (A.4) and (A.5) to (A.11) yields

$$\frac{\partial E}{\partial A} = -2\text{P}^T \hat{H} \text{T} + 2(\text{P}^T \text{P})\text{A}(\text{T}^T \text{T}) \quad (\text{A.12})$$

Thus, the gradient will be zero for the particular value of A given by

$$A = (\text{P}^T \text{P})^{-1} \text{P}^T \hat{H} \text{T} (\text{T}^T \text{T})^{-1} \quad (\text{A.13})$$

as long as matrices $\text{P}^T \text{P}$ and $\text{T}^T \text{T}$ are nonsingular.

APPENDIX B

— CLOSED-FORM SUMMATION OF A COSINE SERIES

Closed form expression of a series comprising N generalized cosine terms, for any real α and any real $\beta \neq 2m\pi$ (m an integer), is given by [41]

$$\sum_{k=0}^N \cos(\alpha + k\beta) = \frac{\cos(\alpha + \frac{N}{2}\beta) \sin(\frac{N+1}{2}\beta)}{\sin(\frac{\beta}{2})} \quad (\text{B.1})$$

APPENDIX C

A MATRIX INVERSION LEMMA

If Γ , and Ψ , and A are three matrices with Γ being nonsingular and the orders of the three matrices being compatible to form the composite matrix $(\Gamma + \Psi A)$, then the inverse of this composite matrix can be expressed as [42]

$$(\Gamma + \Psi A)^{-1} = \Gamma^{-1} - \Gamma^{-1} \Psi (I + A \Gamma^{-1} \Psi)^{-1} A \Gamma^{-1} \quad (C.1)$$

where I is a compatible identity matrix.